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Designing a STEM-Based Learning Trajectory on Tangent Lines to Parabolas Using a Football Context for Pre-Service Teachers

Tika Dwi Nopriyanti[1], Zulkardi[2*], Ratu Ilma Indra Putri[3], Nyimas Aisyah[4]
[1,2,3,4]Universitas Sriwijaya, Palembang, Indonesia
Correspondence should be addressed to Author: zulkardi@gmail.com

Abstract

Understanding the concept of a tangent line to a parabola is an important topic but often a challenge for prospective mathematics teacher students when learning emphasizes procedures over conceptual understanding. This research aims to design a STEM-based learning trajectory (Hypothetical Learning Trajectory) to enhance students' mathematical flexibility skills. The research method uses design research with three stages: initial design, learning experiment (pilot and full implementation), and retrospective analysis. The research subjects consisted of pre service mathematics teacher students from two classes. The learning path includes three STEM activities: exploring the crossbar challenge video in soccer, algebraic analysis and visualization using Desmos, and constructing and verifying tangents analytically and graphically. Data were obtained through classroom observation, interviews, worksheets, and videos, and were analyzed qualitatively using triangulation and the constant comparison method. The results show that students move from intuitive reasoning toward formal understanding thru transitions in contextual, symbolic, and graphical representations. Revising the HLT results in a Local Instructional Theory (LIT) that strengthens students' thinking flexibility and conceptual understanding. The results suggest that subsequent studies could refine and implement this LIT across diverse mathematical topics or STEM-integrated learning environments to examine its broader impact on students' representational and reasoning flexibility.

Keywords: STEM, Tangent Lines, Learning Trajectory, Football Context, Design Research

Abstrak

Pemahaman tentang konsep garis singgung pada parabola merupakan topik yang penting namun sering menjadi tantangan bagi mahasiswa calon guru matematika ketika pembelajaran lebih menekankan pada prosedur daripada pemahaman konseptual. Penelitian ini bertujuan untuk merancang lintasan belajar berbasis STEM (Hypothetical Learning Trajectory) guna meningkatkan kemampuan fleksibilitas matematis mahasiswa. Metode penelitian yang digunakan adalah design research yang terdiri atas tiga tahap: desain awal, eksperimen pembelajaran (uji coba awal dan implementasi penuh), serta analisis retrospektif. Subjek penelitian terdiri atas mahasiswa calon guru matematika dari dua kelas. Lintasan belajar yang dikembangkan meliputi tiga aktivitas berbasis STEM, yaitu mengeksplorasi video crossbar challenge dalam sepak bola, melakukan analisis aljabar dan visualisasi menggunakan Desmos, serta membangun dan memverifikasi garis singgung secara analitik dan grafis. Data diperoleh melalui observasi kelas, wawancara, lembar kerja mahasiswa, dan rekaman video, kemudian dianalisis secara kualitatif dengan teknik triangulasi dan metode perbandingan konstan. Hasil penelitian menunjukkan bahwa mahasiswa mengalami pergeseran dari penalaran intuitif menuju pemahaman formal melalui transisi representasi kontekstual, simbolik, dan grafis. Revisi terhadap HLT menghasilkan Local Instructional Theory (LIT) yang memperkuat keluwesan berpikir dan pemahaman konseptual mahasiswa. Hasil ini juga menunjukkan bahwa penelitian lanjutan dapat menyempurnakan dan mengimplementasikan LIT tersebut pada berbagai

topik matematika atau lingkungan pembelajaran berbasis STEM untuk menelaah dampak yang lebih luas terhadap fleksibilitas representasional dan penalaran mahasiswa.

INTRODUCTION

Background

The topic of tangent lines to parabolas is not merely a subtopic within analytic geometry but a crucial component that underpins the development of students' analytical and representational reasoning ((Biza, 2021; Kondratieva & Bergsten, 2021) Through this topic, learners are guided to understand the relationship between the quadratic curve and its algebraic representation—an essential foundation for
50 advanced mathematical thinking. Previous studies have emphasized that coordinating visual and algebraic reasoning in quadratic functions enables students to generalize and connect symbolic forms with graphical meaning (Wilkie, 2024), while curriculum analyses highlight that many learning difficulties stem from the lack of such representational coherence (Reid O'Connor & Norton, 2024). Integrating geometric visualization and algebraic structure within quadratic contexts supports higher-order reasoning and flexible mathematical thinking (Wilkie, 2022). According to Kemendikbudristek (2022), the teaching of analytic geometry encompasses quadratic curves (parabolas and circles), and the gradient of tangent lines serves as a key concept for building representational connections and solving real-world problems such as analyzing ball trajectories, designing parabolic antennas, and calculating signal reflections (Santos-Trigo et al., 2024; Suci Wulandari et al., 2024).

Problem Identification

Despite its importance, students often struggle to understand the tangent line to a parabola conceptually. While some can apply analytical formulas or derivatives, many fail to relate them to the geometric meaning or real-world context of the parabola (García-García et al., 2025a). They tend to
34 perceive the tangent merely as "a line that touches," without understanding the condition of equal
2 gradients or the role of the tangency point. Only a small percentage of students can define the tangent line as the limit of secant lines approaching the point of contact (Borji et al., 2024; García-García et al.,
18 2025a). These misconceptions are often reinforced by instructional practices that focus on procedural calculation rather than conceptual understanding (Rahayuningsih et al., 2025)

Theoretical Foundation

A major challenge in understanding parabolic concepts lies in the limited use of real-world contexts that connect abstract mathematics to concrete experiences. Mathematical flexibility—the ability to creatively and adaptively employ multiple representations and strategies to solve problems—develops when instruction encourages transitions between geometric, algebraic, and contextual forms (Bolat &
33 Arslan, 2024; Hickendorff et al., 2022; Jóelsdóttir et al., 2024). Studies show that representational
21 transitions enhance conceptual understanding and problem-solving (Ayyıldız Altınbaş et al., 2025;
Sproesser et al., 2022). The integration of authentic contexts is particularly effective in teaching tangent lines, as it allows students to link visual and algebraic representations (Bos & Wigmans, 2023; Kondratieva & Bergsten, 2021). A relevant real-world example is soccer, where the parabolic trajectory of a kicked ball can be analyzed to determine whether it will strike the goal crossbar—a context that vividly illustrates the geometric meaning of tangency (Santos-Trigo et al., 2024). Similarly, Nopriyanti et al., (2025) emphasizes that embedding real-world phenomena in geometry instruction enhances conceptual reasoning and engagement by allowing students to visualize mathematical ideas meaningfully. Thus, authentic contexts act not only as motivational tools but also as conceptual bridges that deepen understanding of tangency, slope, and the relationship between curves and equations.

Research Gap

Existing studies on tangent lines have largely focused on diagnosing misconceptions (Biza & Zachariades, 2010; García-García et al., 2025b) or examining STEM approaches for improving conceptual understanding (Portillo-Blanco et al., 2025; Roehrig et al., 2021). However, few have integrated these two strands—conceptual development and STEM-based design—within a coherent instructional framework. Moreover, research in the Indonesian context indicates a need for pre-service teachers to experience meaningful learning that connects formal mathematics with contextual reasoning, rather than relying solely on procedural methods (Hartini, 2020; Rahayuningsih et al., 2025)

Research Contribution

4 This study addresses these gaps by developing a STEM-based Hypothetical Learning Trajectory (HLT)
for tangent lines to parabolas, designed through a design research approach. The HLT aims to guide students from intuitive (visual) reasoning to formal (symbolic) understanding through technological exploration. It also fosters mathematical flexibility by integrating design-based and real-world

problem-solving activities that require students to shift between graphical, analytical, and numerical strategies. The framework aligns with the five principles of integrated STEM learning—cross-
56 disciplinary integration, real-world problems, design-based learning, inquiry, and collaboration (Portillo-Blanco et al., 2025). Within this framework, activities such as the *"crossbar challenge"* in soccer serve as engaging contexts for modeling parabolic trajectories and determining tangent lines through geometric analysis. Ultimately, this study contributes a Local Instructional Theory (LIT) that bridges contextual STEM experiences with conceptual understanding, advancing both theoretical and practical
4 insights into the development of mathematical flexibility among pre-service teachers.

METHOD

Research Approach

48 This study employs a design research approach of the validation study type as outlined by (Gravemeijer
39 and Cobb (2006). Design research was chosen because it provides a systematic framework for
52 designing, implementing, and revising learning trajectories through iterative cycles of design and
53 analysis. The primary aim is to produce a Learning Trajectory (LT) that can serve as a guide for lecturers or other educators in teaching similar concepts across different contexts. According to Gravemeijer and
12 Cobb (2006) and Bakker (2018), the design research process consists of three main phases.

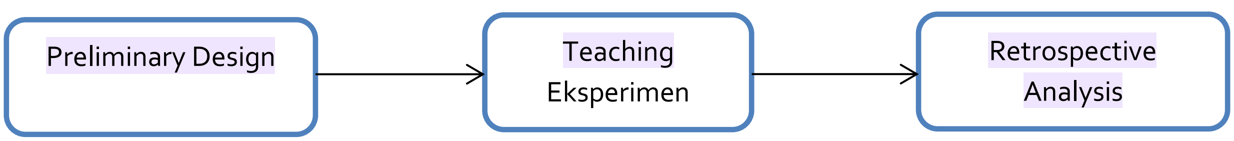


Figure 1. , Proses design research Gravemeijer and Cobb (2006) and Bakker (2018)

59 In addition to Gravemeijer and Bakker, this study also draws on the principles of Hypothetical Learning
1 Trajectories (Simon, 1995), which emphasize the relationship between learning goals, learning activities, and the anticipated progression of students' understanding (the hypothetical learning process). Within this framework, the HLT on tangent lines to parabolas was designed to guide pre-service teacherss from context-based intuitive understanding (e.g., the crossbar challenge in soccer) toward the formal representation of the tangent line equation of a parabola, supported by STEM-based learning.

The STEM approach employed in this study follows the integration framework described by (Bybee
8 11 ,2013; Roehrig et al., 2021; Portillo-Blanco et al. 2025), which highlights five principles: (1) cross-

disciplinary integration, (2) the use of real-world problems, (3) design-oriented solutions, (4) evidence-based inquiry, and (5) collaboration. Integrating these principles into the HLT is expected to foster
7 mathematical flexibility, as defined by Hickendorff et al. (2022), namely the ability to creatively and adaptively shift across strategies and representations in solving mathematical problems.

Research Subject

7 The subjects of this study consisted of 22 seventh-semester prospective mathematics teachers in the pilot experiment who had completed the Geometry course, and 27 third-semester prospective mathematics teachers in the teaching experiment who were currently taking the same course. The selection of these subjects was based on academic and theoretical considerations in design research, which emphasizes the importance of involving participants who possess relevant prior knowledge or are in the process of developing it (Gravemeijer & Cobb, 2006) In this context, students who had taken or were taking the Analytic Geometry course were deemed appropriate because they had been introduced to the fundamental concepts of curves, slopes, and tangent lines—knowledge essential for engaging meaningfully in the designed learning trajectory.

Data Collection

35 Data collection was carried out through interviews, observations, student activity sheets, and
1 documentation in the form of video recordings. Interviews were conducted with the course lecturer to
17 review the learning process and to gain insights into students' ways of thinking. Observations focused on
17 the implementation of instruction and student interactions during classroom discussions. Student activity sheets were developed based on the HLT designed with STEM principles as the foundation and were then used to evaluate the progress of the learning activities. After the instructional activities, students were given a test to assess their learning achievements from the designed tasks, which
10 subsequently served as a basis for revising the HLT activities. In addition, documentation in the form of
1 photographs and video recordings was conducted during both the pilot experiment and the teaching experiment to capture the learning process throughout the study..

Data Analysis

Data were analyzed retrospectively with reference to the HLT using interviews, classroom observations, student activity sheets, and documentation. The qualitative analysis emphasized validity and reliability. Validity was ensured through triangulation (interviews, observations, documents, and video recordings) and process traceability, allowing readers to follow the learning trajectory and reasoning behind each conclusion. Reliability was maintained through detailed procedural descriptions and cross-interpretation across data sources.

Data from activity sheets, interviews, and observations were directly linked to each phase of the initial HLT. Students' responses and strategies provided evidence for confirming or revising the conjectures of each activity, while patterns from the data guided adjustments to better align the HLT with students' actual learning processes.

This approach is consistent with Creswell & Creswell (2018) who emphasize that qualitative validity ensures interpretations reflect participants' experiences, and reliability concerns consistency in reporting. It also aligns with Miles, Huberman, and Saldaña (2014), whose framework of data reduction, display, and verification was realized through systematic triangulation. To ensure trustworthiness,
16 Lincoln and Guba's (1985) criteria—credibility, transferability, dependability, and confirmability—were applied so that the conclusions remained credible and accountable.

RESULTS AND DISCUSSION

Preliminary Design

1 At this stage, the researcher developed an initial HLT to be used in the learning process. The researcher then engaged in discussions with the Geometry course lecturer regarding the initial HLT design. The activities designed for the tangent line topic consisted of three activities and employed the Desmos application as a tool for visual verification. Desmos is a web-based mathematical visualization platform that enables students and teachers to explore mathematical concepts through interactive graphs, digital manipulatives, and real-time modeling activities. Beyond functioning as a graphing calculator, Desmos has evolved into an interactive learning medium that supports student-centered and inquiry-based learning approaches. Its use in mathematics instruction promotes students' flexibility of thought by allowing them to explore multiple representations of concepts, select diverse problem-solving strategies, and enhance conceptual understanding through interactive visualization (Alabdulaziz &
8 Higgins, 2025; Chien et al., 2025; Christie et al., 2025; Jumaniyazov et al., 2025).

Teaching Eksperimen

55 The STEM-based learning activities were developed based on the initially designed Hypothetical Learning Trajectory (HLT) and aimed to foster students' mathematical flexibility. The pilot experiment served as the initial implementation stage to test the designed HLT and examine its feasibility in guiding students' conceptual development. Findings from this stage provided critical feedback for

revising the HLT and resulted in an Adjusted Learning Trajectory (ALT), which was then used in the subsequent teaching experiment. Learning activities for the tangent line material were divided into three main stages, each designed to help students deepen their understanding of tangent lines and
45 apply related concepts to real-world situations. The relationship between the STEM-based learning
1 activities and the tangent line material is presented in Table 1.

Table 1. The Relationship Between STEM-Based Learning and the Concept of Tangent Lines

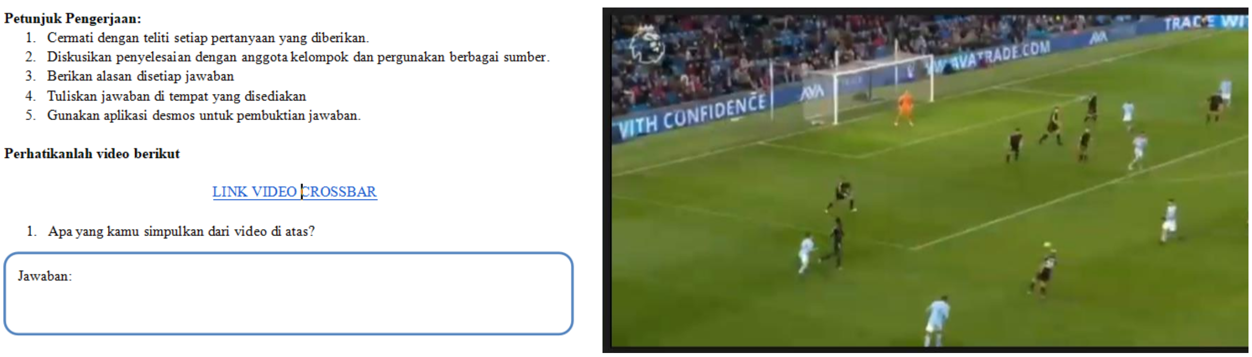
	Learning Path	Learning Activity	Tangent Lines equation of Parabola
19 42	Activity 1 Observing and analyzing the crossbar challenge video	1. Watching a video about the crossbar challenge. 2. Concluding that the point of tangency is the meeting point between the ball's trajectory and the crossbar, which occurs at exactly one point.	Understanding the concept of a tangent point in the context of the crossbar challenge.
29	Activity 2 Determining the tangent line equation of a parabola in the context of the ball's trajectory	1. Presenting the problem of a ball's trajectory in the form of a parabolic equation. 2. Determining whether the ball will hit the crossbar by analyzing the trajectory equation. 3. Students find the equation of the tangent line at the meeting point of the ball and the crossbar. 4. Students sketch the tangent line and the ball's trajectory using the Desmos application.	Determining the tangent line equation of a parabola.
5	Activity 3 Predicting the tangent line equation	1. Students open the Desmos application and plot a parabola using its general equation. 2. Students select a point on the parabola curve and discuss how to determine the slope of the tangent line analytically.	Determining the tangent line equation from the graph or from a given point.

Activity 1: Observing and Analyzing the Crossbar Challenge Video

12 The first activity in this learning trajectory begins with the context of the Crossbar Challenge, where a player attempts to kick the ball so that its trajectory hits the crossbar. This phenomenon is chosen because it is highly relatable to students' everyday experiences and directly connected to the
41 mathematical idea of a tangent line. From a pedagogical perspective, using the context of a popular sport such as soccer not only enhances learning motivation but also provides a realistic learning experience aligned with the principles of PMRI. Through this contextual exploration, students are guided to observe and analyze the parabolic motion of the ball, recognize the point of contact between

the trajectory and the crossbar as a tangency point, and gradually connect this understanding to the formal equation of a tangent line to a parabola.

From a STEM perspective, the Crossbar Challenge serves as the starting point for integrating science, technology, engineering, and mathematics. Scientifically, students analyze the parabolic motion of the ball influenced by gravity. The technological aspect emerges when students use the Desmos application to visualize the ball's trajectory and verify the tangent point. From the engineering perspective, students design strategies or simulations of how the ball can hit the crossbar. Meanwhile, the mathematical aspect focuses on exploring the tangent point, which in this context appears as the intersection between the parabolic trajectory of the ball and the horizontal line of the crossbar. *Figure 2* illustrates students' activities related to the crossbar challenge.



Petunjuk Pengerjaan:
1. Cermati dengan teliti setiap pertanyaan yang diberikan.
2. Diskusikan penyelesaian dengan anggota kelompok dan pergunakan berbagai sumber.
3. Berikan alasan disetiap jawaban
4. Tuliskan jawaban di tempat yang disediakan
5. Gunakan aplikasi desmos untuk pembuktian jawaban.

Perhatikanlah video berikut

LINK VIDEO CROSSBAR

1. Apa yang kamu simpulkan dari video di atas?

Jawaban:

Figure 2. Students' Activity Display on the Crossbar Challenge

Through this activity, students are encouraged to observe a video of the crossbar challenge and then connect it with geometric concepts. Guiding questions such as "*What are the possible outcomes when a crossbar occurs?*" or "*What geometric term is used to describe the intersection of the ball's trajectory with the goalpost?*" prompt students to develop intuitive understanding before moving on to more formal mathematical representations. In this way, the activity serves as a bridge between real (concrete) experiences and abstract concepts (the tangent line), in line with the goals of STEM-based learning that emphasize the integration of practical experiences with scientific reasoning.

Perhatikanlah video berikut

LINK VIDEO CROSSBAR

1. Apa yang kamu simpulkan dari video di atas?

Jawaban: Dari video crossbar tersebut menunjukkan konsep titik singgung dalam kehidupan sehari-hari, khususnya dalam permainan sepak bola ketika bola mengenai mistar gawang (crossbar). Video tersebut menggambarkan bagaimana bola memantul sesuai dengan garis singggung

2. Kemungkinan apa saja yang bisa terjadi jika terdapat crossbar?

Jawaban: Jika bola mengenai crossbar, ada beberapa kemungkinan :
- Bola memantul ke dalam gawang dan mencetak gol
- Bola memantul keluar lapangan tanpa mencetak gol
- Bola memantul dan jatuh kembali ke lapangan

English Version

1. **What can you conclude from the video above?**
   Answer: The crossbar video shows the concept of tangent lines in everyday life, especially in football when the ball hits the crossbar. The video illustrates how the ball bounces according to the tangent line direction.
2. **What are the possible outcomes if the ball hits the crossbar?**
   Answer: If the ball hits the crossbar, there are several possibilities:
   a. The ball goes into the goal and results in a score
   b. The ball bounces outside the field without scoring
   c. The ball bounces and falls back into the field

Figure 3. Students' Responses to Activity 1

Based on the students' responses, it can be seen that they understood the mathematical concept of a tangent line through a real-life example in soccer, namely when the ball hits the crossbar. They were able to explain that the point of tangency determines the direction of the ball's rebound, which reflects an accurate conceptual understanding. In addition, the students also demonstrated flexibility in thinking, by recognizing that a single event (the ball touching the crossbar) can lead to multiple possible outcomes— a goal, a miss, or the ball returning to the field. The ability to consider various possibilities rather than being fixated on a single outcome is an important indicator of flexible thinking. This shows that the students not only understood the concept theoretically but were also able to connect it to real situations and think openly about different possibilities. *Figure 3* below presents the students' responses to the subsequent question in Activity 1.

4. Bagaimanakah suatu kejadian yang menyebabkan terjadinya *crossbar* ?

Jawaban: Crossbar terjadi ketika bola yang ditendang memiliki lintasan yang cukup tinggi tetapi tidak melebihi gawang. Ini biasanya terjadi jika tendangan memiliki sudut elevasi yang terlalu tinggi atau kecepatan yang cukup kuat untuk mencapai mistar gawang.

5. Didalam geometri Istilah apa yang digunakan untuk menyebut titik pertemuan antara lintasan bola dan gawang?

Jawaban: Istilah yang digunakan adalah *titik singgung*, yaitu titik di mana bola menyentuh mistar gawang sebelum memantul.

6. Berdasarkan aktivitas yang dilakukan, apa yang dapat kamu simpulkan tentang garis singgung?

Jawaban: Garis singgung adalah garis yang menyentuh suatu kurva di satu titik tanpa memotongnya. Dalam konteks video, titik singgung terjadi ketika bola menyentuh mistar gawang sebelum berubah arah sesuai dengan hukum pantulan.

English Version

4. What kind of situation causes a crossbar to happen?
Answer:
A crossbar occurs when a kicked ball has a trajectory that is high enough but does not exceed the height of the goal. This usually happens if the kick has a steep elevation angle or enough speed to reach the crossbar.
5. In geometry, what term is used to describe the point where the ball's path meets the goalpost?
Answer:
The term used is *tangent point*, which refers to the point where the ball touches the crossbar before bouncing.

40 **6. Based on the activity you did, what can you conclude about a tangent line?**
Answer:
3 A tangent line is a line that touches a curve at exactly one point without crossing it. In this context, the tangent point occurs when the ball touches the crossbar before changing direction according to the law of reflection.

Figure 4. Responses of Students in Activity 1

13 Based on the students' responses, it can be seen that they demonstrated a good understanding of the
26 tangent line concept, particularly in explaining that the point of tangency is the point where the ball
touches the crossbar before bouncing, in accordance with the principle of reflection. In addition, the students also displayed flexibility by explaining that a crossbar event may occur due to various factors, such as the angle of elevation and the speed of the kick. This reflects flexible thinking, as the students were able to view a single phenomenon from multiple possible causes and relate it contextually to mathematical concepts.

14 After conducting the pilot experiment and analyzing the findings during the learning process as well as
the students' work in completing Activity 1, the researcher held discussions with the lecturer to revise
1 the HLT so that it became more focused and aligned with the learning objectives. *Table 2* presents a
comparison between the HLT and ALT in Activity 1.

Table 2. Comparison between HLT and ALT in Activity 1

Learning Activity	HLT	ALT
1. Watching and analyzing a video about the crossbar challenge. 2. Concluding that the point of tangency is the meeting point between the ball's trajectory and the crossbar, which occurs at exactly one point.	1. Students are able to analyze a video about the crossbar challenge and relate it to the concepts of the parabola and the tangent line. 2. Students are able to conclude that the point of tangency is the intersection between the ball's trajectory and the crossbar, which occurs at exactly one point	Students are able to explicitly connect the phenomenon with geometric concepts: 1. "What term is used in geometry to refer to the intersection point between the ball's trajectory and the goalpost?" 2. "If the point on the crossbar forms a straight line, then how is that line positioned relative to the ball's trajectory?"

Activity 2: Tangent Line Equation of a Parabola in the Context of Ball Trajectory

In the design experiment phase of the second activity, the learning was focused on applying the context
2 of soccer to understand the concept of a tangent line to a parabola. This activity was designed so that

15 students would not only be able to determine the equation of a tangent line, but also interpret real-life situations on the field through mathematical representations. With this approach, students were encouraged to connect contextual problems—such as the trajectory of a ball hitting the crossbar or the position of a photographer's camera—with algebraic and calculus skills in determining tangent points and the gradient of a parabola.

In the process, the integration of STEM elements became evident: science through the analysis of the ball's trajectory as a parabolic motion phenomenon; technology through the use of the Desmos application for visualization and verification; engineering through the mathematical modeling of contextual events; and mathematics through the application of derivatives and systems of equations to determine the tangent line of a parabola.

The results of this pilot experiment provided an initial picture of how students' learning trajectories developed as they connected sports phenomena with analytic geometry concepts, while also demonstrating the potential of STEM integration in facilitating more meaningful learning. *Figure 4* below illustrates the second activity given to the students.

Tujuan
Mahasiswa dapat:
1. Menentukan persamaan garis singgung parabola
2. Mensketsa garis singgung parabola

Petunjuk Pengerjaan:
1. Cermati dengan teliti setiap pertanyaan yang diberikan.
2. Diskusikan penyelesaian dengan anggota kelompok dan pergunakan berbagai sumber.
3. Berikan alasan disetiap jawaban
4. Tuliskan jawaban di tempat yang disediakan
5. Gunakan aplikasi desmos untuk pembuktian jawaban.

Perhatikan dan analisis masalah berikut lalu jawablah pertanyaan-pertanyaannya !

Seorang pemain sepak bola menendang bola sehingga lintasannya mengikuti persamaan parabola berikut:
$y = -\frac{1}{4}x^2 + x$
di mana x adalah jarak horizontal (meter), dan y adalah ketinggian bola (meter).
Mistar gawang berada pada posisi x = 6 meter dan ketinggian y = 2.5 meter.

1. Sketsalah lintasan bola yang terjadi!

Jawaban

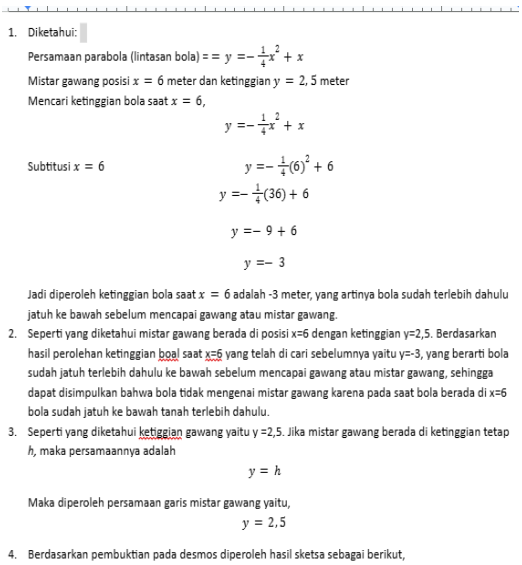
2. Prediksilah yang terjadi lintasan bola mengenai mistar gawang, lalu buktikan jawabanmu dengan perhitungan atau penalaran yang logis!

Jawab:

Figure 5. Worksheet 2

In the implementation of the second activity, students were engaged in analyzing the trajectory of a soccer ball modeled by a parabolic equation. The contextual problem addressed was the situation in
31 which a ball is kicked toward the goal with the possibility of hitting the crossbar. Students were asked to sketch the trajectory of the ball and then determine mathematically whether the ball touches the crossbar. Through this series of tasks, students not only carried out algebraic and calculus procedures but also developed a conceptual understanding that the tangent line is a mathematical representation of the real interaction between the ball's trajectory and another object on the field.

Based on the results of this pilot experiment, several important findings were obtained regarding students' thinking patterns, problem-solving strategies, and the difficulties they encountered when 46 connecting contextual phenomena with mathematical concepts. These findings will be presented in the following section to provide a more detailed picture of the students' learning trajectories in this activity.



1. Diketahui:

   Persamaan parabola (lintasan bola) = = $y = -\frac{1}{4}x^2 + x$

   Mistar gawang posisi $x = 6$ meter dan ketinggian $y = 2,5$ meter

   Mencari ketinggian bola saat $x = 6$,

   $$y = -\frac{1}{4}x^2 + x$$

   Subtitusi $x = 6$

   $$y = -\frac{1}{4}(6)^2 + 6$$

   $$y = -\frac{1}{4}(36) + 6$$

   $$y = -9 + 6$$

   $$y = -3$$

   Jadi diperoleh ketinggian bola saat $x = 6$ adalah -3 meter, yang artinya bola sudah terlebih dahulu jatuh ke bawah sebelum mencapai gawang atau mistar gawang.

2. Seperti yang diketahui mistar gawang berada di posisi x=6 dengan ketinggian y=2,5. Berdasarkan hasil perolehan ketinggian boal saat x=6 yang telah di cari sebelumnya yaitu y=-3, yang berarti bola sudah jatuh terlebih dahulu ke bawah sebelum mencapai gawang atau mistar gawang, sehingga dapat disimpulkan bahwa bola tidak mengenai mistar gawang karena pada saat bola berada di x=6 bola sudah jatuh ke bawah tanah terlebih dahulu.

3. Seperti yang diketahui ketiggian gawang yaitu y =2,5. Jika mistar gawang berada di ketinggian tetap *h*, maka persamaannya adalah

   $$y = h$$

   Maka diperoleh persamaan garis mistar gawang yaitu,

   $$y = 2,5$$

4. Berdasarkan pembuktian pada desmos diperoleh hasil sketsa sebagai berikut,

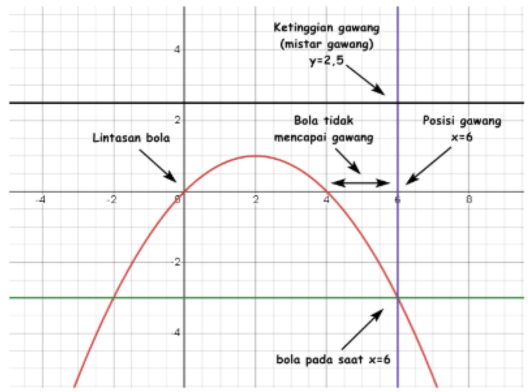


Figure 6. Student Group Discussion Results for Activity 2

3 Based on the analysis of Activity 2, the students have understood that the tangent line to a
3 parabola is a line that touches the curve at exactly one point. Although in this case the ball did not hit the crossbar, the students still demonstrated an important understanding: to determine a tangent line, it is necessary to identify the point of tangency on the parabola that is parallel to a given line (in this case, the horizontal line of the crossbar). Thus, their understanding has progressed from mere numerical calculations to conceptual skills in connecting real-world phenomena, algebraic representations, and graphical visualizations. After completing Activity 2, the researcher conducted 47 interviews with the students to examine the effectiveness of the activity. The following section presents the results of the student interviews.

Lecturer : *"What do you think about the problem in the second activity regarding the ball's trajectory toward the goal?"*

Student A : *"In my opinion, the problem was interesting because it was directly connected to soccer. But when we calculated it, it turned out the ball did not touch the crossbar, so we could only conclude that the ball fell earlier."*

2 **Lecturer** : *"Did that help you understand the concept of the tangent line to a parabola?"*

6 **Student A** : *"It did help, but not completely. Because we didn't get the chance to find the tangent line that actually touches the parabola at the point on the goal. So it felt like something was missing, as if the concept did not fully appear."*

Lecturer : *"What do you think could be added to make the understanding of the tangent line clearer?"*

Student B : *"Maybe there should be an additional problem where the ball's trajectory really touches the crossbar. If we had that case, we could prove the tangent line, not just stop at the conclusion that the ball did not reach."*

Lecturer : *"How about the problem-solving methods—was there a particular method that helped you more?"*

Student B : *"If we used only one method, sometimes we weren't sure. But when we were asked to try both derivatives and elimination, we could compare the results. That made us more confident that the tangent line equation was correct."*

Based on the results of the pilot experiment and student interviews, it was found that the initial problem in cctivity 2 only produced a ball trajectory that did not touch the crossbar, preventing students from experiencing the process of determining a tangent line in a real situation. To address this, discussions with the model lecturer led to a revised scenario in which the ball's trajectory actually touched the crossbar, allowing students to authentically explore the tangent line by identifying the exact point of tangency. The revision also aimed to enrich students' problem-solving strategies through two complementary approaches: the elimination method, involving a quadratic–linear system, and the derivative method for determining the slope of the tangent. Students were guided to compare both methods to see how they converge on the same tangency point—the former emphasizing algebraic relationships and the latter the geometric interpretation of slope. This reflective comparison helped students perceive the methods as conceptually interconnected rather than procedural alternatives, thereby strengthening their learning trajectory and enhancing the quality of STEM integration. Table 3 presents the comparison between the HLT and ALT in activity 2.

Table 3. Comparison between HLT and ALT in Activity 2

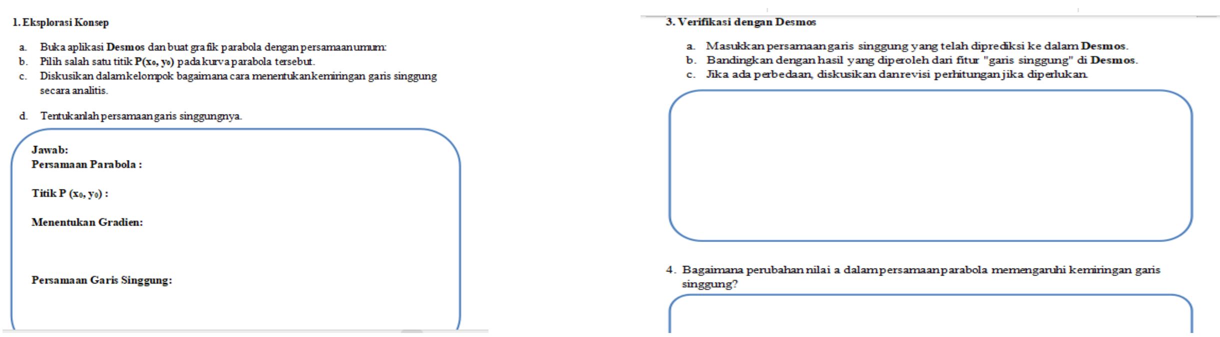
Learning Activity	HLT	ALT
1. Presenting the problem of the ball's trajectory in the form of a parabolic equation. 2. Determining whether the ball will hit the crossbar by analyzing the trajectory equation. 3. Students find the equation of the tangent line at the intersection point of the ball and the crossbar. 4. Students sketch the tangent line and the ball's trajectory using the Desmos application.	1. Students determine the ball's trajectory from the given equation. 2. They attempt to determine whether the ball hits the crossbar by using substitution and simple manual calculations. 3. Students determine the equation of the tangent line. 4. Students use the Desmos application for verification.	1. Students first predict whether the ball will hit the crossbar, then verify it through mathematical calculations and graphs. 2. Students are asked to determine the equation of the tangent line at the crossbar point as a representation of the trajectory's direction at that point. 3. Students use two explicit approaches. 4. Students compare the results from both approaches and

draw conclusions.
5. Students use Desmos to connect algebra and geometry.

5

Activity 3. Determining the Equation of a Tangent Line from a Graph or from a Given Point

The third activity aimed to deepen students' understanding of slope and tangent line equations on a parabola through the integration of technology and analytical approaches. Using the Desmos application, students explored the graph of a parabola, selected a specific point, and determined the tangent line equation analytically. This process exemplified STEM-based learning: science in interpreting slope as a derivative, technology through Desmos visualization and verification, engineering in designing predictive methods from point data, and mathematics in applying differential techniques. Students verified their results using Desmos' tangent feature and engaged in group discussions to evaluate accuracy and construct arguments across visual and symbolic representations. The activity trained students to think flexibly by shifting between representations and formulating coherent justifications, thereby linking procedural understanding with conceptual reasoning through technology-supported exploration. Figure 7 illustrates Activity 3 on tangent lines to a parabola.


1. Eksplorasi Konsep

a. Buka aplikasi **Desmos** dan buat grafik parabola dengan persamaan umum:
b. Pilih salah satu titik **P(x_0, y_0)** pada kurva parabola tersebut.
c. Diskusikan dalam kelompok bagaimana cara menentukan kemiringan garis singgung secara analitis.

d. Tentukanlah persamaan garis singgungnya.

Jawab:
Persamaan Parabola :

Titik P (x_0, y_0) :

Menentukan Gradien:

Persamaan Garis Singgung:

3. Verifikasi dengan Desmos

a. Masukkan persamaan garis singgung yang telah diprediksi ke dalam **Desmos**.
b. Bandingkan dengan hasil yang diperoleh dari fitur "garis singgung" di **Desmos**.
c. Jika ada perbedaan, diskusikan dan revisi perhitungan jika diperlukan.

4. Bagaimana perubahan nilai a dalam persamaan parabola memengaruhi kemiringan garis singgung?

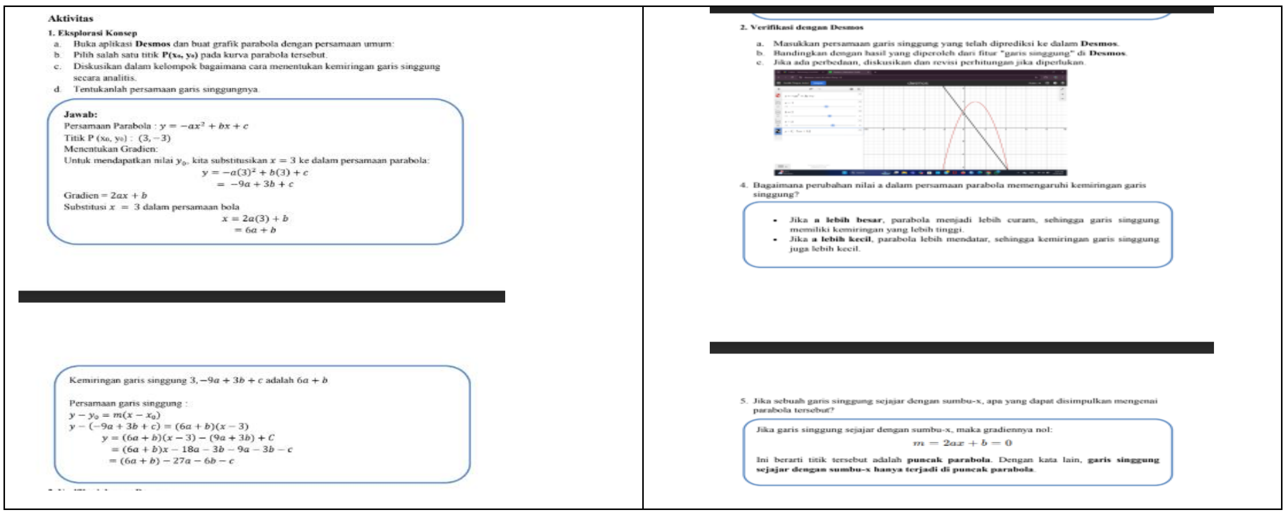
Figure 7. Activity 3

In the third activity, the design focused on developing students' conceptual understanding of slope and
2
the equation of a tangent line to a parabola through a STEM-based, inquiry-oriented approach.
4
Students were guided to use the Desmos application to represent the graph of a parabola, select a point on the curve, and then predict the tangent line equation analytically based on the principles of derivatives.

The science aspect was reflected in the understanding of derivatives as rates of change, technology in the use of Desmos as an exploratory and verification tool, engineering in designing predictive strategies
5
to determine the slope of the line, and mathematics in the symbolic formulation of the tangent line

equation. The predicted results were then compared with direct visualizations through the tangent line feature in the application, enabling students to revise their approaches reflectively.

Group discussions encouraged them to evaluate the accuracy of their calculations, construct coherent mathematical arguments, and compare different problem-solving strategies. This activity aimed to foster students' mathematical flexibility through transitions across representations (graphical,
30 symbolic, technological), while also deepening their understanding of the relationship between differential geometry and functional visualization in the context of technology-enhanced learning. The following section presents the results of student group discussions for the third activity.



Aktivitas

1. Eksplorasi Konsep

a. Buka aplikasi **Desmos** dan buat grafik parabola dengan persamaan umum:
b. Pilih salah satu titik **P(x₀, y₀)** pada kurva parabola tersebut.
c. Diskusikan dalam kelompok bagaimana cara menentukan kemiringan garis singgung secara analitis.
d. Tentukanlah persamaan garis singgungnya.

Jawab:
Persamaan Parabola : $y = -ax^2 + bx + c$
Titik P (x_0, y_0) : $(3, -3)$
Menentukan Gradien:
Untuk mendapatkan nilai y_0, kita substitusikan $x = 3$ ke dalam persamaan parabola:
$y = -a(3)^2 + b(3) + c$
$= -9a + 3b + c$
Gradien = $2ax + b$
Substitusi $x = 3$ dalam persamaan bola
$x = 2a(3) + b$
$= 6a + b$

[redacted]

Kemiringan garis singgung $3, -9a + 3b + c$ adalah $6a + b$

Persamaan garis singgung :
$y - y_0 = m(x - x_0)$
$y - (-9a + 3b + c) = (6a + b)(x - 3)$
$y = (6a + b)(x - 3) - (9a + 3b) + C$
$= (6a + b)x - 18a - 3b - 9a - 3b - c$
$= (6a + b) - 27a - 6b - c$

[redacted]

2. Verifikasi dengan Desmos

a. Masukkan persamaan garis singgung yang telah diprediksi ke dalam **Desmos**.
b. Bandingkan dengan hasil yang diperoleh dari fitur "garis singgung" di **Desmos**.
c. Jika ada perbedaan, diskusikan dan revisi perhitungan jika diperlukan.

4. Bagaimana perubahan nilai a dalam persamaan parabola memengaruhi kemiringan garis singgung?

- Jika **a lebih besar**, parabola menjadi lebih curam, sehingga garis singgung memiliki kemiringan yang lebih tinggi.
- Jika **a lebih kecil**, parabola lebih mendatar, sehingga kemiringan garis singgung juga lebih kecil.

[redacted]

5. Jika sebuah garis singgung sejajar dengan sumbu-x, apa yang dapat disimpulkan mengenai parabola tersebut?

Jika garis singgung sejajar dengan sumbu-x, maka gradiennya nol:
$m = 2ax + b = 0$
Ini berarti titik tersebut adalah **puncak parabola**. Dengan kata lain, **garis singgung sejajar dengan sumbu-x hanya terjadi di puncak parabola**.

Figure 8. Students' Responses in the Third Activity

Analysis of students' responses revealed a growing understanding of the tangent line as a mathematical object linking geometric and analytical representations. Through exploring points on a parabola and using derivatives to determine slopes, students recognized that the gradient at a given point is defined by the local properties of the function, not chosen arbitrarily. This understanding
2 reflects a shift from viewing the tangent merely as a "touching line" to conceiving it as the limit of secant lines approaching the tangency point, consistent with the formal definition in calculus.
57 Moreover, students' ability to construct tangent line equations from a point and slope indicated mastery of linear equations and their analytic justification through derivatives. Verification using Desmos strengthened the link between symbolic and visual reasoning, as students compared predicted and actual graphs, reflected on discrepancies, and deepened their conceptual insight—understanding that mathematical validity involves both procedural accuracy and geometric meaning.

In addition, students' responses to conceptual questions about the effect of the coefficient a in the
13 parabola equation on the slope of the tangent line indicated a functional understanding of the role of
49 parameters in altering the properties of the curve and the rate of change at a given point. They also

understood that a tangent line parallel to the x-axis occurs at the vertex of the parabola, showing that they had internalized the idea that a zero derivative marks the extreme point of a function. After the third activity, the researcher conducted interviews with students to examine whether the STEM-based learning that took place supported their understanding of the tangent line topic.

Lecturer *: After completing the third activity, how did you find the process of determining the tangent line to a parabola?*
Student A *: In my opinion, the process was quite clear. We started from the parabola graph in Desmos, then chose a point on the curve and determined the gradient using derivatives. From there, we were able to construct the tangent line equation.*
Lecturer *: How did Desmos help you in understanding this?*
Student C *: Desmos was very helpful. We could immediately see whether the line we predicted truly touched the curve at a single point. If it didn't match, we knew we had to check our calculations again. The visualization made us more confident about the concept.*
Lecturer *: Do you think this STEM-based learning made the lesson more engaging?*
Student A *: Yes, because we weren't just calculating; we also analyzed scientifically, used technology, and discussed together to solve the problem. We even tried other scenarios to see how changes in the curve affected the tangent line.*
Student B *: I also felt that we learned more comprehensively. There was the science aspect when discussing curve changes, technology through Desmos, mathematics in the formulas, and engineering when designing solution strategies.*
Lecturer *: In that case, do you think this activity helped you understand the tangent line concept better compared to previous lessons?*
Student C *: Much more helpful, Sir. We didn't just memorize formulas; we truly understood how the tangent line works. And we could explain why the results turned out that way, not just present the final answer.*

The reflections drawn from the findings during the learning process, student interviews, and activity results were discussed in an FGD with the model lecturer. It was concluded that some students had not yet fully understood the role of coefficients in influencing the shape of the curve. Therefore, in the discussion process, it was necessary to guide students in determining the appropriate values of *a*, *b*, and *c* to produce a clear parabola graph.

Table 4. Comparison between HLT and ALT in Activity 3

Learning Activity	HLT	ALT
1. Plotting a parabola using the general equation. 2. Selecting a point on the parabola curve and discussing how to determine the slope of the tangent line analytically.	1. Students plot a parabola using the general equation. 2. Students select a point on the parabola curve and discuss how to determine the slope of the tangent line analytically.	1. Students first predict whether the ball will hit the crossbar, then verify it through mathematical calculations and graphs. 2. Students are asked to determine the equation of the tangent line at the crossbar point as a representation of the trajectory's direction at that point.

	3. Students use two explicit approaches. 4. Students compare the results of both approaches and draw conclusions. 5. Students use Desmos to connect algebra and geometry.

Retrospective Analysis

The retrospective analysis focused on evaluating the alignment between the designed Hypothetical Learning Trajectory (HLT) and the Actual Learning Trajectory (ALT) observed during the pilot and teaching experiments. Overall, the findings revealed both consistencies and necessary revisions that contributed to the refinement of the Local Instructional Theory (LIT).

In Activity 1, students were able to relate the crossbar challenge video to the concept of tangent lines. Their responses demonstrated intuitive recognition that the tangent point determined the ball's rebound direction. This confirmed the initial HLT prediction that contextual phenomena could bridge everyday experiences with abstract geometric reasoning. However, classroom discussions also indicated the need to include more guiding questions so that students could explicitly articulate the connection between the observed context and the mathematical concept. This revision contributed to the LIT by emphasizing the pedagogical value of contextual anchoring as the entry point for developing conceptual understanding.

In Activity 2, the initial problem produced a ball trajectory that did not touch the crossbar. Although students correctly concluded that the ball fell before reaching the goal, they did not experience the process of deriving a tangent line from a real contact point, limiting the realization of the HLT's conceptual goals. The revised task introduced a scenario where the ball's trajectory actually touched the crossbar, enabling students to construct and compare tangent equations using both system elimination and derivative methods. This comparison strengthened the LIT by illustrating how dual-solution strategies can promote representational flexibility and deepen conceptual coherence.

In Activity 3, students demonstrated significant progress in transitioning from visual to symbolic representations. They recognized the tangent line not merely as "a line that touches" but as the limit of secant lines approaching a single point, consistent with the formal definition in calculus. The integration of Desmos facilitated verification and reflection, allowing students to test their predictions graphically and analytically. However, some students still struggled to interpret the influence of

4 coefficients (a, b, c) on the shape of the parabola, which led to the inclusion of additional scaffolding tasks in the refined LIT to support symbolic reasoning.

Taken together, the retrospective analysis shows that each iterative revision—from contextual anchoring (Activity 1), dual-method exploration (Activity 2), to representational integration (Activity 3)—contributed directly to the formulation of a contextually grounded LIT. The cycle of testing, feedback, and refinement between the HLT and ALP thus produced a coherent theoretical framework that integrates STEM principles to foster both conceptual understanding and mathematical flexibility.

Discusion

20 This study reveals that a STEM-based learning trajectory effectively supports the development of prospective mathematics teachers' mathematical flexibility in understanding the concept of the
6 tangent line to a parabola. These findings align with prior research emphasizing that flexibility emerges when learners engage with multiple representations and connect contextual, symbolic, and graphical reasoning (García-García et al., 2025a; Hickendorff et al., 2022). By embedding realistic contexts, mathematical modeling, and technology integration, the learning trajectory not only fostered deeper conceptual understanding but also provided evidence of how iterative design can bridge procedural and conceptual knowledge. This suggests that incorporating STEM principles within a structured HLT framework can serve as a powerful pedagogical model for cultivating adaptive and connected
25 mathematical thinking among pre-service teachers.

In the first activity, the use of the Crossbar Challenge video proved effective as a contextual entry point. This finding complements the perspective of Santos-Trigo et al., (2024), who emphasized the importance of dynamic contexts, by demonstrating that a familiar soccer phenomenon can stimulate students' geometric intuition toward the parabola. Students' recognition of the tangent point as the unique intersection between the ball's trajectory and the crossbar confirmed the initial conjecture that contextual experiences can activate intuitive reasoning, forming a foundation for formal exploration.

The second activity built upon this intuition by guiding students to analytically determine the tangent line through two complementary approaches: the system elimination and derivative methods. The inclusion of a revised scenario where the ball's trajectory actually touched the crossbar enabled students to compare these methods critically. Unlike previous studies that treated strategy selection as
38 independent choices (Hickendorff et al., 2022; Jóelsdóttir et al., 2024), this research demonstrated that flexibility can be fostered through reflection on how both strategies converge toward the same

mathematical idea. This reinforces the novelty of the design, in which problem-solving diversity serves as a scaffold for representational coherence.

The third activity consolidated learning by transitioning from visual reasoning to formal symbolic
3 representation. Students began to interpret the tangent line not only as a touching line but as the limit of secant lines—addressing a misconception highlighted by Biza & Zachariades (2010). With the support of Desmos, they linked the coefficients of the parabola's equation with the slope of its tangent, illustrating a more integrated understanding between algebraic form and geometric meaning. This use of technology provided empirical evidence that visualization tools can accelerate conceptual progression from concrete to abstract reasoning.

Viewed collectively, the three activities functioned as an interconnected sequence within the HLT—each stage preparing the conceptual groundwork for the next. The contextual exploration (Activity 1) evoked intuition, the analytical dual-method investigation (Activity 2) deepened reasoning, and the technological formalization (Activity 3) solidified understanding. Together, they formed a coherent learning trajectory that advanced students' transition from intuitive to formal mathematical thinking. The study's main contribution lies in demonstrating how integrating STEM principles—science through analyzing parabolic motion, technology through Desmos, engineering through problem design, and mathematics through formalization—within a unified HLT framework can promote both conceptual understanding and mathematical flexibility, representing a state-of-the-art approach to geometry
8 learning in 21st-century education (Portillo-Blanco et al., 2025; Rahayuningsih et al., 2025)

Implication of Research

22 The findings of this study provide several important implications for various stakeholders in
37 mathematics education. For future researchers, the Learning Trajectory (LT) developed in this study can serve as a reference framework for designing and refining learning trajectories in other mathematical topics, such as calculus, geometry, or algebra, which likewise demand flexibility in shifting between representations. The design research methodology, with its iterative cycles, also opens opportunities for comparative studies across different educational levels and diverse cultural contexts.

For practitioners—lecturers or teachers—the STEM-based learning trajectory designed in this study offers a practical model for integrating real-world contexts, such as the soccer crossbar challenge, with the mathematical concept of tangent lines. The structured activities encourage students to progress from intuitive reasoning to formal understanding through multi-representational tasks, thereby providing educators with concrete strategies to foster students' flexibility in thinking and depth of conceptual understanding.

51 For policymakers and curriculum developers, this study underscores the importance of embedding STEM principles and realistic contexts within teacher education programs. By supporting the integration of technology (e.g., Desmos) and inquiry-based learning, educational policies can better prepare pre-service teacherss to deliver meaningful and flexible mathematics learning that aligns with 21st-century competency demands.

Overall, the implications of this study affirm that a STEM-based learning trajectory is not only relevant for enhancing students' conceptual understanding of tangent lines, but also holds potential to enrich
7 teaching practices, strengthen curriculum design, and open new directions for research. Thus, the results of this study are expected to make a tangible contribution to the development of theory, practice, and policy in mathematics education.

Limitation

28 This study has several limitations that should be acknowledged. In terms of participants, the study involved only 49 prospective mathematics teachers from two classes, so the findings cannot yet be generalized broadly. In terms of content, the focus was limited to the concept of tangent lines to parabolas, meaning that the effectiveness of the designed STEM-based learning trajectory may not necessarily apply to other mathematical topics without adaptation. In terms of context, the learning activities were carried out in classrooms with the support of specific technology (Desmos), which may not fully represent school conditions with limited access to technology. In terms of time, the research was conducted within a single semester, which does not capture the long-term impact on students'
24 development of mathematical flexibility. In terms of design, this study employed a design research approach of the validation study type, emphasizing internal validity through the testing of the HLT, but it did not compare the effectiveness of this learning trajectory with other approaches in an experimental manner.

CONCLUSION

This study highlights a coherent pattern showing that a STEM-based learning trajectory, developed through a design research approach, can serve as an effective framework for fostering mathematical flexibility and conceptual understanding among prospective mathematics teachers. The integration of contextual phenomena, symbolic reasoning, and technology demonstrated how authentic contexts can bridge intuitive and formal thinking within mathematics learning. The core contribution of this research lies in formulating a Local Instructional Theory (LIT) that connects STEM principles with the progressive development of geometric reasoning, offering both theoretical and practical implications for mathematics teacher education. This LIT can be adapted to other calculus and geometry topics—such as derivatives of trigonometric functions, optimization problems, or the study of circle and ellipse tangency—by maintaining the integration of contextual modeling, technology-based visualization, and analytical reasoning. Moreover, its implementation in secondary school settings could provide students with meaningful experiences that connect mathematical abstraction to real-world phenomena, thereby promoting flexible and connected mathematical thinking. Future research may further explore these adaptations to evaluate their effectiveness in diverse educational contexts.

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