UJME 13 (1) 2024: 11-27
UNNES JOURNAL OF MATHEMATICS EDUCATION
https://journal.unnes.ac.id/journals/ujme
ISSN: 2252-6927 (print); 2460-5840 (online)

Using the History of Circles and Parabolic Segment Areas as a Learning Alternative in Integral

Layli Rahmania Yuwono[a,*], Janny Lindiarni[b], Rizal Afgani[b]

[a] *Faculty of Mathematics and Natural Science, Bandung Institute of Technology, Bandung 40132, Indonesia*
[b] *Analysis and Geometry Research Group, Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia*

* *E-mail address: layliry@gmail.com*

ARTICLE INFO

Article history:
Received 17 October 2023
Received in revised form 30 December 2023
Accepted 3 March 2024

Keywords:
Integral; area; circle, parabola segment; Eudoxus; Archimedes

Abstract

This article will present some classic problems in the Ancient Greece period: the ratio of the areas of two circles problem solved by Eudoxus and the area of a parabola segment problem solved by Archimedes. These problems can be used as alternative teaching resources to give the students an early understanding of the integral concept. This article focuses on finding alternatives for teaching integral material through theorems and historical understanding without calculus knowledge. This study used a systematic literature review method to analyze the mathematical content and the historical influences on their problem-solving methods. The literature sources were indirect sources such as journals, books, and other written literature. The results show that Eudoxus' principle has been a special limit problem since the period, helping solve the ratio of the areas of two circles problem, and there has been a special case of infinite geometric series solving the area of parabolic segment problem. This article gives some recommendations for the teachers at the end of the article, on how to give a representation of the propositions discussed in this article to the students so the students can understand the connections between the prior area problem (in which the area is bounded by its line segments) and the integral concept which will be learned.

1. Introduction

The concept of integral in mathematics is one of the essential and fundamental concepts in calculus learning. It is a calculus concept that developed from the need to solve area and volume problems (Bauldry, 2009a). One of the learning difficulties experienced by students in the application of integral concepts for area problems is difficulty in understanding the basic concepts of integral (Susilo et al., 2019a). Significant indicators of conceptual understanding are students' ability to present and explain different mathematical situations for various purposes and connect them with other related concepts. Conceptual Understanding is expected to be possessed by every learner. If a student has good concept understanding, then he/she can synthesize the important mathematical ideas learned and understand the types of contexts used (Kilpatrick, 2001).

According to the Merdeka Curriculum, calculus elements are found in the Learning Outcomes at the senior high school level. It means that before entering the undergraduate level, calculus learning learned at the high school level is the initial foundation for students in learning calculus. Previous knowledge can not only be a very strong support for students to receive further learning, but it can also lead students to build conceptual understanding which, if not appropriate, can be a barrier in learning (Donovan, 2005).

Known mathematical problems solved by the concept of integral refer to the concept of integral discussed by Leibniz (1646 - 1716). The integral concept was also contributed by the definition of limit discussed by Weierstrass (1815-1897). Little is known about the early classical forms of solving problems

To cite this article:
Yuwono, L. R., Lindiarni, J., & Afgani, R. (2024). Using the History of Circles and Parabolic Segment Areas as a Learning Alternative in Integral. *Unnes Journal of Mathematics Education*, *13*(1), 11-27. https://doi.org/10.15294/vw82bv85

of area and volume of geometric figures before the emergence of modern limit definition formulations. Previous methods used by mathematicians before Weierstrass in solving calculation problems of area and volume of geometric figures underpinned the need for a limit definition that aided the concept of integrals. In the ancient Greek era, mathematicians such as Eudoxus (408 - 355 BC) and Archimedes (287 - 212 BC) illustrated the solution to the problem of the area bounded by curves. Both of them are based on the idea of the integral concept. Yet, there are differences in their mathematical practices in solving it

The methods they used at that time did not use the justification of the integral concept yet. But, it is undeniable that the ideas they used in solving the problems underlie the emergence of the integral concept that has been discovered.

The reference records of previous mathematical discoveries provide information about the development of mathematical knowledge and procedures, the use of mathematics, and the types of important problems solved at that time (Katz, 1998a). The discoveries of previous mathematicians can show that there are still obstacles in solving mathematical problems such as the tendency to ignore the failure of the method used or the difficulty to understand how a theorem applies in different contexts. The Awareness of the diversity of mathematical practices can invite us to explore further the mathematical concepts learned.

The previous experience of mathematicians in solving problems regarding areas before they could be solved with the concept of integral in modern times can be an illustration for students to understand that various methods of solving for different problem contexts were used before the concept of integral emerged. Hence, by using historical understanding in integral learning, students are able to understand the different contexts of problems that can be summarised with modern integral ideas.

Mathematics teaching through materials developed with the history of mathematics can increase students' learning achievement levels (Kaygin et al., 2011a). One of the contributions of applying mathematics history in mathematics learning can also provide a detailed understanding of mathematical concepts and theorems (Sukarani, 2022).

Referring to the above background, the research focuses on finding alternatives to teach the integral concept through theorems and historical understanding without calculus knowledge. The discussed problems include the problem of the area of geometric figures solved by mathematicians in Ancient Greece (400-200 BC). Topics discussed include the problem of the area of geometric figures solved by mathematicians in Ancient Greece (400-200 BC). Moreover, the problem studied is also a problem that only requires prerequisite material that has been passed by high school students. The analysis results in this article can contribute to providing alternatives that allow mathematics teachers to provide students with an initial understanding of the background of the integral concept. It is expected that with an understanding of the historical side, students can understand the basic concept of integral which further includes not only the concept, but also includes an understanding of problem-solving methods with the concept of integral. The analysis results in this article can contribute to providing alternatives that allow mathematics teachers to provide students with an initial understanding of the background of the integral concept. Hopefully, with an understanding of the historical side, students can understand the basic concept of integral which subsequently includes not only the concept, but also includes an understanding of problem-solving methods with the concept of integral.

2. Methods

This research was conducted using the systematic literature review method. It is the process of identifying, reviewing, and evaluating existing research in a focused and interesting topic area, by presenting relevant research questions. In accordance with the purpose of this research, the method was used to present an alternative to the previous mathematical problems that underpinned the emergence of the integral concept. Hence, the focus of this research is to review the literature that conveys the ideas of Eudoxus and Archimedes regarding the solution of area problems as the background of the modern integral concept.

The literature sources used are indirect sources. It presents historical facts conveyed by a third party (Y. Topolski, 2012). Besides, the classification of literature used is written literature. Included in this classification are sources taken from journals, books, or other written literature. Hence, in this research, the intent of the information conveyed in the literature used was reviewed.

The initial stage of the research was to define the questions to be answered and agreed on the research objectives. Based on the objectives chosen, good quality literature is selected and answers the question that has been set. Searching for sources to obtain related data or documents by studying secondary data sources regarding the research problem.

The second stage of the research is the verification. In this stage, the verification is checking the relevant information listed in the source. The researcher checked the theorems presented from the sources by proving them using techniques that had been done at that time and techniques that were already known during the research if needed. The technique used during the figures solved the problem could be different from what is known at the time of this research. It is due to the continuous development of mathematical knowledge. Furthermore, the proof conducted is also reviewed whether the theorem researched is acceptable with high school student's knowledge level. It was done to achieve the objectives that have been stated in the introduction.

The final stage is the interpretation of the facts obtained from the sources. In this research, the interpretation is done by explaining descriptively or descriptive explanation. This is a descriptive explanation that does not include an explanation of the causes and origins of related information that is difficult to find the original truth (Topolski, 2012). The description was chosen because the researcher will focus on the figure's mathematical understanding of the problem and the problem-solving method used. It is this description that is useful as an alternative to conveying the background of the integral concept by the teacher in explaining the integral concept.

3. Results & Discussions

3.1. The Area Ratio of Two Circles and The Eudoxus Principle

It has been stated in the background that this article presents an alternative to conveying an initial understanding of the integral concepts to students from a historical perspective. The review of area problems in geometry solved in the ancient Greek period in this article is presented through the proof of the circle area theorem by Eudoxus and the parabolic segment by Archimedes.

The number concept and magnitude used in this research refer to the concepts that prevailed in Ancient Greece.

Definition 1. *Numbers are natural numbers.*

Magnitude is the measure of the length of line segments, angle magnitude, area, and volume of a shape. In this article the length of line segment AB with endpoints A and B is denoted as $|AB|$. Also the area of triangle ΔCDE is denoted as $|\Delta CDE|$. For other areas of polygon with vertices $A_1, A_2, A_3, \ldots$ and A_N for some $N = 1,2,3, \ldots$, is denoted as $|A_1A_2A_3 \ldots A_N|$.

Antiphon (480 - 411 BC) initiated the Method of Exhaustion. It is a method of determining the area of a circle by approximating it with the area of the regular polygon (stated in Definition 2) that inscribes it (Ball, 1960).

Definition 2. *Given a point* O *and a number* $\mathrm{N} > 2$. *Select points* $\mathrm{A_i}$ *with* $\mathrm{i} = 1,2,3, \ldots, \mathrm{N}$ *such that for every* i *holds* $|\mathrm{OA_i}| = \mathrm{r}$ *for some* r *and* $|\mathrm{A_1A_2}| = |\mathrm{A_2A_3}| = |\mathrm{A_3A_4}| = \cdots |\mathrm{A_{N-1}A_N}| = |\mathrm{A_NA_{N+1}}|$. *The Union line segments is called an* N*-sided regular polygons, or can be abbreviated as a regular polygon. Furthermore, the length* r *is called the radius of the polygon.*

Prior to the proof of Eudoxus' principle, there was a theorem about two areas of regular polygons with the same number of sides and different radii already known.

Lemma 1. *Let N be a number. Given regular polygons* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *which it's sides sum are N. If* $\mathcal{P}_1$ *has polygon radius* $\mathrm{r_1}$ and $\mathcal{P}_2$ *has polygon radius* $\mathrm{r_2}$*, then the area of* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *is*

$$\frac{\mathrm{a}(\mathcal{P}_1)}{\mathrm{a}(\mathcal{P}_2)} = \frac{\mathrm{r}_1^2}{\mathrm{r}_2^2}.$$

Proof of Lemma 1. Let a regular polygon $\mathcal{P}_1$ has midpoint O_1 and radius r_1. Let also that regular polygon $\mathcal{P}_2$ has midpoint O_2 and radius r_2. $\mathcal{P}_1$ dan $\mathcal{P}_2$ has N sides.

Take an arbitrary points A_1 and A_2 on $\mathcal{P}_1$ such that $|O_1 A_1| = |O_1 A_2| = r_1$. Let $\Delta A_1 O_1 A_2$ has a height along $t_1 = kr_1$ and the base A_1A_2 with the length $|A_1A_2| = \ell_1 = qr_1$ for a magnitude of the length k and q.

Take such arbitrary points B_1 and B_2 on $\mathcal{P}_2$ that $|O_2B_1| = |O_2B_2| = r_2$. Let $\Delta B_1 O_2 B_2$ has the base B_1B_2. Consider that $\mathcal{P}_2$ also an N sided polygon, so it has height along $t_2 = kr_2$ and $|B_1B_2| = \ell_2 = qr_2$.

Since both $\mathcal{P}_1$ and $\mathcal{P}_2$ have N sides, thus $\frac{r_1}{\ell_1} = \frac{r_2}{\ell_2}$ and $\frac{r_1}{t_1} = \frac{r_2}{t_2}$. As a consequence, if $r_2 = cr_1$ for a number c, then $\ell_2 = c\ell_1$ and $t_2 = ct_1$.

Consider that a rectangle area with the length ℓ_1 and the width t_1 has twice area of the triangle $\Delta A_1O_1A_2$. Let the area of $\Delta A_1O_1A_2$ is equal with $a(\Delta A_1O_1A_2)$, then $a(\Delta A_1O_1A_2) = \frac{1}{2}\ell_1 t_1 = \frac{1}{2}qr_1\, kr_1 = \frac{1}{2}q_1 kr_1^2$. The same applies to triangle $\Delta B_1O_2B_2$ if $a(\Delta B_1O_2B_2) = \frac{1}{2}\ell_2 t_2 = \frac{1}{2}qr_2\, kr_2 = \frac{1}{2}qk\,(r_2)^2$.

Accordingly

$$\frac{a(\mathcal{P}_1)}{a(\mathcal{P}_2)} = \frac{Na(\Delta A_1\, O_1A_2)}{Na(\Delta B_1O_2B_2)} = \frac{N\frac{1}{2}\, qr\,(r_1)^2}{N\frac{1}{2}\, qr\,(r_2)^2} = \frac{r_1^2}{r_2^2}.$$

In proving Antiphon's statement, other than the knowledge of Lemma 1, Eudoxus used his principle in Theorem 1.

Theorem 1 : Prinsip Eudoxus. *Let* M_0 *and* ε *are magnitudes with* $\varepsilon < \mathrm{M}_0$. *If the sequence* $\mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3, \ldots$ *meets* $\mathrm{M}_1 < \frac{1}{2}\mathrm{M}_0$, $\mathrm{M}_2 < \frac{1}{2}\mathrm{M}_1$, $\mathrm{M}_3 < \frac{1}{2}\mathrm{M}_2$,... *then there is a number* N *such that* $\mathrm{M}_\mathrm{N} < \varepsilon$.

Theorem 1 is a special case of the familiar modern concept of limits. Understanding about the limit in it leads to understanding the limit in terms of geometry. Theorem 1 applies to every kind of magnitude, including the area of a circle magnitude.

Definition 3. *Given a point* O *and length magnitude* r. *The Union of all points such that* $|\mathrm{OA}| = \mathrm{r}$ *is called a circle with the center point* O *and the radius* r.

Definition 4. *Given a circle* $\mathcal{C}$ *with the area* $\mathrm{a}(\mathcal{C})$ *and a regular polygon* $\mathcal{P}$ *with the area* $\mathrm{a}(\mathcal{P})$. $\mathcal{P}$ inscribes $\mathcal{C}$ *if and only if every angle points of* $\mathcal{P}$ *is in* $\mathcal{C}$.

Theorem 2. *Let* $\mathcal{C}$ *is a circle. For every* ε *such that* $0 < \varepsilon < \mathrm{a}(\mathcal{C})$ *there is a number* N *for which the area of the regular polygons* $\mathcal{P}_\mathrm{N}$ *inscribing* $\mathcal{C}$ *satisfies* $\mathrm{a}(\mathcal{C}) - \mathrm{a}(\mathcal{P}_\mathrm{N}) < \varepsilon$.

Proof : Let a circle $\mathcal{C}$ has area $a(\mathcal{C}) = M_0$. Take any ε such that $0 < \varepsilon < a(\mathcal{C})$. Choose points A, B, C and D on $\mathcal{C}$ so that $ABCD$ is a 4 side regular polygon. Let $\mathcal{P}_1 = ABCD$. Define

$$M_1 = M_0 - a(\mathcal{P}_1). \quad (1)$$

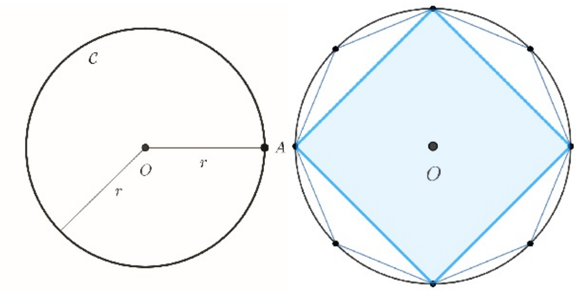

Figure 1. Circle inscribed by regular polygons.

Then select points E, F, G and H on $\mathcal{C}$ so that $AEBFCGDH$ is a 8 sided regular polygon. Let $\mathcal{P}_2 = AEBFCGDH$, then

$M_2 = M_0 - a(\mathcal{P}_2)$. (2)

Using the same process, $\mathcal{P}_{k+1}$ can be obtained from $\mathcal{P}_k$ for each number k. Thus for each number it is obtained $M_k = M_0 - a(\mathcal{P}_k)$.

Select points A, B, C, D on $\mathcal{C}$ so that $ABCD$ is a 4 sided regular polygon. Let $\mathcal{P}_1 = ABCD$. Select points A', B', C', D' so that A is the midpoint of line segment $A'D'$, B is the midpoint of line segment $A'B'$, C is the midpoint of line segment $B'C'$, and A is the midpoint of line segment $C'D'$. Let the area of $\mathcal{P}_1' = A'B'C'D'$. Note that $a(\mathcal{P}_1) = \frac{1}{2}a(\mathcal{P}_1')$ and $a(\mathcal{P}_1') > M_0$. Therefore

$M_0 < 2a(\mathcal{P}_1)$. (3)

Based on (1) and (3), obtained

$$M_1 < \frac{1}{2} M_0.$$

Choose points E, F, G, H on $\mathcal{C}$ such that $AEBFCGDH$ is a 8 sided regular polygon. Also Select points I and J so that line segment IJ is parallel to the line segment DA and H is the midpoint of the line segment IJ. As $|IH| = |HJ|$ and $|DI| = |JA|$ and $\Delta ADH, \Delta ABE, \Delta BCF, \Delta CDG$ are congruent, then,

$M_1 < 2(\mathcal{P}_2 - \mathcal{P}_1)$. (4)

Based on (1), (2), and (4) obtained

$$M_2 < \frac{1}{2} M_1.$$

If $\mathcal{C}$ is being reinscribed with regular polygons $N = 2^{k-1}$ $sided\ for\ each\ k = 3,4,5,\ldots$, the we get

$$M_{k-1} < 2(\mathcal{P}_k - \mathcal{P}_{k-1}).$$

Consequently obtained that

$$M_1 < \frac{1}{2}M_0, M_2 < \frac{1}{2}M_1, M_3 < \frac{1}{2}M_2, \ldots \blacksquare$$

It is proved that the remains of the area that are consecutive to each process fulfilling $\mathrm{M}_1 < \frac{1}{2}\mathrm{M}_0$, $\mathrm{M}_2 < \frac{1}{2}\mathrm{M}_1$, $\mathrm{M}_3 < \frac{1}{2}\mathrm{M}_2$,...

This implies that according to Theorem 1, it holds that there is a number k such that $M_k < \varepsilon$. Select $N = k$. $M_N = M_0 - a(\mathcal{P}_N) = a(\mathcal{C}) - a(\mathcal{P}_N) < \varepsilon$ is obtaied. ■

It has been known that Antiphon's statement can be proved (see Theorem 1). For any selected magnitude ε, there will be a regular polygon that can inscribe the circle so that the difference between the area of the circle and the regular polygon is less than ε.

Theorem 2 explains that there is a link between the concept of limit and the concept of integral. Furthermore It will be reviewed how the method of exhaustion and the Eudoxus principle can produce the ratio of two circles with different radius magnitudes formula.

Theorem 3. *If circle* C_1 *has the radius* r_1 *and circle* C_2 *has the radius* r_2, *then*

$$\frac{a(C_1)}{a(C_2)} = \frac{r_1^2}{r_2^2}.$$

Proof : Suppose $\frac{a(C_1)}{a(C_2)} < \frac{r_1^2}{r_2^2}$ is *equivalent* to $a(C_2) > \frac{r_2^2}{r_1^2}a(C_1)$. Let $S = \frac{r_2^2}{r_1^2}a(C_1)$. Therefore $S < a(C_2)$.

Select $0 < \varepsilon < a(C_2) - S$. Select also a regular polygon $\mathcal{P}_2$ *of side* N *with* $N = 2^{k+1}$ for a $k = 1,2,3, \ldots$ whose area $a(\mathcal{P}_2)$ inscribes the C_2 such that it satisfies $a(C_2) - a(\mathcal{P}_2) < \varepsilon < a(C_2) - S$. As a result

$$a(\mathcal{P}_2) > S. \quad (5)$$

Select a regular polygon $\mathcal{P}_1$ the N *side whose area is* $a(\mathcal{P}_1)$ so that it inscribes C_1. Since $\mathcal{P}_1$ and $\mathcal{P}_2$ have the same number of sides, then according to Lemma 1 $\frac{a(\mathcal{P}_1)}{a(\mathcal{P}_2)} = \frac{r_1^2}{r_2^2} = \frac{a(C_1)}{S}$ is obtained.

Since $a(C_1) > a(\mathcal{P}_1) > 0$, then $S > a(\mathcal{P}_2)$ is contradictory to (5). Thus, the supposition $\frac{a(C_1)}{a(C_2)} < \frac{r_1^2}{r_2^2}$ is incorrect. ■

3.2. The Parabolic Segment Area by Archimedes

Stepping ahead from the circle, Archimedes solved a problem related to one of the sections of conics, the parabola. He calculated the area of a parabolic segment. The methods used were still similar to those of Eudoxus. Archimedes inscribed the empty spaces of the parabolic segment with triangles.

Parabolic segment is one type of plane resulting from a conic section. In this aricle, cones and parabolic segments are defined according to the definition given by Apollonius (240 BC - 190 BC) of Perga.

Definition 5. Let A *is a point and* $\mathcal{C}$ *is a circle. The set of points on the lines which joining* A *with points on the circle* $\mathcal{C}$ *forms a surface called a cone. A double cone is an equal and opposite cone and meets at point* A.

Definition 6. *An Apex is a point where a double cone meets.*

Definition 7. *Given a cone and an infinite plane* $\mathcal{P}$. *The intersection of the cone with* $\mathcal{P}$ *is called as axial triangle if and only if the diameter line of the cone base and the apex point are in* $\mathcal{P}$.

Theorem 4. *Given a cone with base* $\mathcal{C}$. *The intersection of the cone with all the planes parallel to the circle of the cone base* $\mathcal{C}$ *forms a circle.*

Proof : Given a cone with apex point A and base $\mathcal{C}$. Let $\mathcal{C}$ has a midpoint O. Take an arbitrary point D on $\mathcal{C}$.

Given an infinite plane $\mathcal{P}$. Let $\mathcal{C}'$ is an arbitrary intersection $\mathcal{P}$ with the cone such that $\mathcal{P}$ is parallel to $\mathcal{C}$. As the consequence there is a point o in $\mathcal{C}'$ and a line segment AO. Select point d in $\mathcal{C}'$ such that line segment od is parallel to line segment OD.

Since triangles ΔAod and ΔAOD are congruent, then $\frac{Ao}{AO} = \frac{Ad}{AD}$. Consequently $\frac{od}{OP} = \frac{Ao}{AO}$.

$\mathcal{C}$ is a circle, so that for each D in $\mathcal{C}$ the length of segment OD is constant. Hence the length of space od is also constant for every d in $\mathcal{C}$'. Thus, $\mathcal{C}$' is a circle. ■

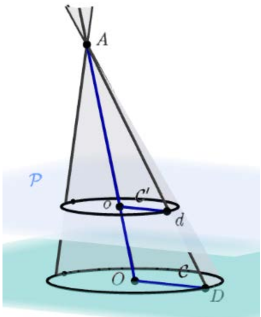

Figure 2. Bases of a cone.

Below is a discussion of the parabola as a conic section in accordance with the definition of Apollonius.

Definition 8. *Given a cone and an infinite plane $\mathcal{P}$. The intersection of the cone and $\mathcal{P}$ such that one of the lines forming the surface of the cone is parallel to $\mathcal{P}$ is called as a parabola. The Parabola bounded by the line at the base of the cone is called as parabolic segment.*

Definition 9. *Given a parabola. Select arbitrary conic base $\mathcal{C}$. The line segment connecting the points of the parabolic section with $\mathcal{C}$ is called as the base of the parabolic segment*

Definition 9 explains how is the difference between a parabola with a parabolic segment. It leads the understanding to the next concept to be determined which is the area of a parabolic segment. Whereas Definition 9 leads us to the generalization of the concept for arbitrary parabolic segment base as per Definition 9. In this article, if a parabolic segment with base line segment Qq and vertex P is given, then it is denoted as the parabola segment QPq.

To form the inscribed triangle, it is necessary to determine the third vertex other than the intersection points between the base of the parabolic segment and the parabola. Next, take the intersection point of the parabola with a line that intersects each midpoint (see Definition 10) of the parabolic segment base.

Definition 10. *V is the midpoint of line segment QQ' if and only if V in on line QQ' and $|QV| = |VQ'|$.*

Lemma 2. *Given a parabolic segment with base DE and midpoint M. Let QQ' be the points on the parabola such that QQ' is parallel to DE with V as the midpoint of QQ'. Then draw a line ℓ which connecting M with V. If R and R' are points on the parabola such that RR' is parallel to DE, then the midpoint of the line segment RR' is on line ℓ*

Proof : Let DE is at the base of cone $\mathcal{C}$. Let also that point A is the apex of the cone.

Select points B and C on $\mathcal{C}$ so that the line segment BC is through M and perpendicular to DE. Consequently ΔBAC is an axial triangle.

Since the plane ΔBAC is perpendicular to DE and the line segment QQ' is parallel to DE, then the point $in\ the\ plane$ ΔBAC. As the consequence, the line ℓ is also in the plane ΔBAC.

Let R and R' are points on the parabola so that the line segment RR' is parallel to DE. Consequently the plane ΔBAC is also perpendicular to RR'. Hence the midpoint line of RR' is also in line ℓ. ■

Based on Lemma 2 the line ℓ can intersect the parabola at a single point.

Definition 11. *Given a parabolic segment with base DE. Let point M is the midpoint of segment DE. Take an arbitrary point Q in the parabola. There is a point Q' in the parabola such that the line segment QQ' is parallel to DE. Also Let point V is the midpoint of QQ'. The line which connecting M with V is called diameter line . The intersection of the diameter with the parabola at a point called vertex.*

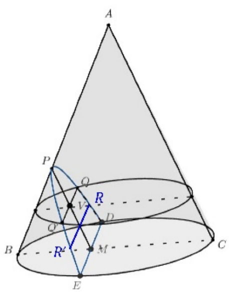

Figure 3. Parabola resulted from the intersection of P and the cone.

It has been known from Definition 11 that there is a single vertex for every parallel segment base in a parabola. Hence, the initial inscribed triangle with its angle points being the intersection points of the segment base with the parabola and the vertex can be formed. Since there are still remaining region, the determination of the vertices of the inscribed triangle will be further explained as follows.

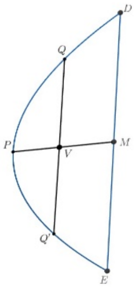

Figure 4. Diameter line at the parabolic segment.

Definition 12. *Given a parabolic segment with base DE. Let M is the midpoint of DE. Hereinafter DM or ME is called ordinate.*

Definition 13. *Given a parabolic segment with base DE. Let M is the midpoint of DE. Abscissa is a line segment on the diameter such that it connects point M with the vertex of the parabola.*

Theorem 5. *Given a Cone BAC. Let the parabolic segment DPE is the intersection of BAC. Let also M is a point in DE so that line segment PM is the diameter od the parabolic segment which parallel to one side of the axial triangle ABC, call it as line segment AC. Let Q is an arbitrary point of the parabolic segment and V is a point in line segment PM so that line segment QV is the ordinate of diameter PM. If line segment PL is perpendicular to the line segment PM such that*

$$\frac{|PL|}{|PA|} = \frac{|BC|^2}{|BA|\cdot|AC|}, \quad (6)$$

then

$$|QV|^2 = |PL| \cdot |PV|.$$

Proof : Let the cone ABC with the points B and C are on the base of the cone such that line segment BC is the cone base diameter.

Let DPE is the parabolic segments resulting from the intersection of cone ABC so that DE is the parabolic segment base and P is on the line AB.

Let M is the midpoint of line segment DE. As a consequence point P and M are on the parabolic segment diameter . Take an arbitrary conic base with points H and K on the conic base such that line segment HK is the diameter and intersects line segment PM at V.

Since the line segment QV is an ordinate,then the line segment QV is parallel to the line segment DE. Consider the circle through HQK. It is obtained that $|QV|^2 = |HV| \cdot |KV|$.

Lets condiser the triangles ΔBPM and ΔBAC. As ΔBPM and ΔHPV are congruent and ΔBAC and ΔHAK are congruent also ,it is obtained $|HV|:|PV| = |BC|:|AC|$ and $\frac{|VK|}{|PA|} = \frac{|BC|}{|BA|}$. Due to this, we obtained $\frac{|QV|^2}{|PV|\cdot|PA|} = \frac{|BC|^2}{|BA|\cdot|AC|}$.

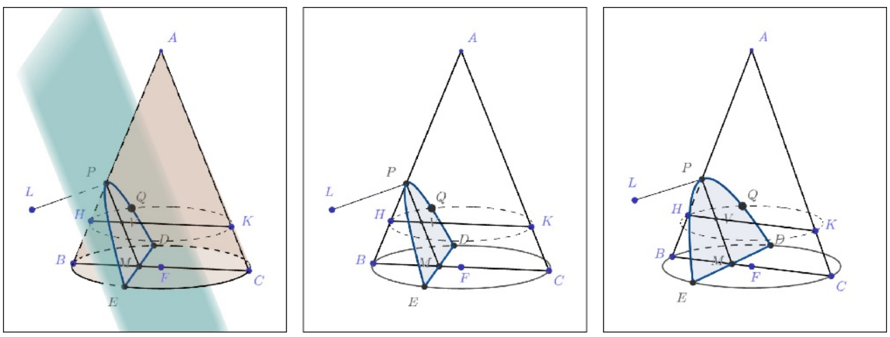

Figure 5. Parabola as one of conic sections.

Since (6) is known , we get $\frac{|QV|^2}{|PV|\cdot|PA|} = \frac{|PL|}{|PA|}$. Thus $|QV|^2 = |PL|\cdot\ |PV|$.

Theorem 6. Let Q is a point on a parabola. Point V is an intersection point of the ordinate through Q with the diameter. Also let point P is the intersection point of the diameter and the parabola, and T is a point on the diameter that lies outside the parabola $|TP| = |PV|$ if and only if line TQ is tangential the parabola at point Q.

Proof : $\Longrightarrow$ Let ℓ is the line TQ. It will be proved that ℓ does not pass inside the parabola. Consequently $\ell$is not a tangent line.

Suppose ℓ passes inside the parabola. Select arbitrary point K on ℓ such that K on the parabola ordinate. Let QV is an ordinate line segment. Select points Q' on the parabola and point V'on the diameter so that the line $Q'KV'$ is parallel to the line segment QV.

Since $|Q'V'| > |KV'|$, then

$$\frac{|Q'V'|^2}{|QV|^2} > \frac{|KV'|^2}{|QV|^2}. \quad (7)$$

$\Delta\ TV'K$ is similar with $\Delta\ TVQ$, so that

$$\frac{|KV'|}{|QV|} = \frac{|TV'|}{|TV|}. \quad (8)$$

Consequently

$$\frac{|Q'V'|^2}{|QV|^2} > \frac{|TV'|^2}{|TV|^2}. \quad (9)$$

Based on the parabola parameters described in Theorem 5, we can get

$$\frac{|PV'|}{|PV|} > \frac{|TV'|^2}{|TV|^2} \quad (10)$$

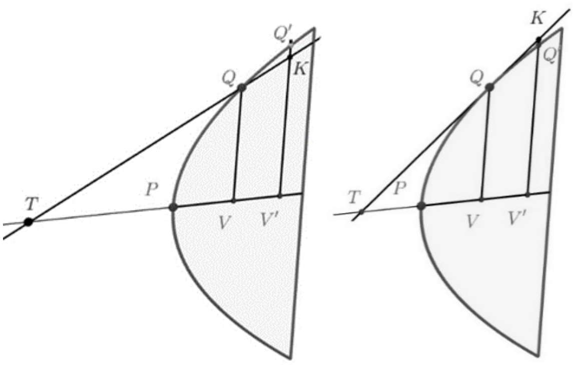

Figure 6. Parabola as one of conic sections.

Equivalent to

$$\frac{4|TP|\cdot|PV'|}{4|TP|\cdot|PV|} > \frac{|TV'|^2}{|TV|^2}. \quad (11)$$

Since $|TP| = |PV|$ dan $|TV| = |TP| + |PV|$, are known, it can be found

$$4|TP|\cdot\ |PV| = 2|TP|\cdot\ 2|PV| = |TV|^2. \quad (12)$$

Consequently

$$4|TP| \cdot |PV'| > |TV'|^2. \quad (13)$$

Yet, $|TP| \neq |PV'|$. Consequently

$$4|TP| \cdot |PV'| < |TV'|^2.$$

Thus, the assumption of the line ℓ passing inside the parabola is incorrect.

$\Leftarrow$ Let the line segment TQ is a line tangent to point Q. Select arbitrary point Q' on the, point V' on the diameter, and point K on the line segment TQ such that there is a line segment $KQ'V'$ parallel to line segment QV.

As the point K is outside the curve such that $|Q'V'| < |KV'|$, then

$$\frac{|Q'V'|^2}{|QV|^2} < \frac{|KV'|^2}{|QV|^2}$$

Using the similar procedure to (10) to (13) obtained

$$\frac{4|TP| \cdot |PV'|}{4|TP| \cdot |PV|} < \frac{|TV'|^2}{|TV|^2}.$$

1. Let $|TP| = |PV'|$, then $4|TP| \cdot |PV'| = |TV'|^2$. Consequently $4|TP| \cdot |PV| > |TV|^2$.
2. Let $|TP| \neq |PV'|$, then $4|TP| \cdot |PV'| < |TV'|^2$. Sonsequnetly $4|TP| \cdot |PV| = |TV|^2$ or $4|TP| \cdot |PV| > |TV|^2$.
   If $|TV| = |TP| + |PV|$, then $4|TP| \cdot |PV| \leq |TV|^2$. Therefore it should be $4|TP| \cdot |PV| = |TV'|^2$. Thus $|TP| = |PV|$. ■

Lemma 3. *Given a parabola with vertex P.let the points Q, R, and R' are on the parabola so that line segment RR' is parallel to the parabola tangent line at point Q. Also let W is a point on the parabola diameter so that line segment RW is the abscissa ordinate of PW. If point F is on the line segment RW so that line QF is parallel to the parabola diameter , then the line QF intersects the line segment RR' at its midpoint.*

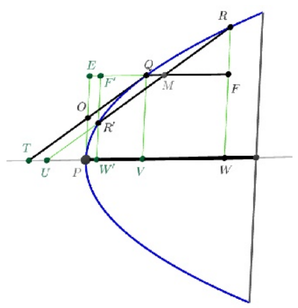

Figure 7. Diameter line of the new parabolic segment.

Proof: Let the line RR' intersects the parabola diameter line at point U and the parabola tangent line at point Q intersects the parabola diameter line at point T. Also Let point E is the intersection line of QF and the line parallel to the ordinate segment through point P. As the consequence, there is point V between P and W so that line segment QV parallel to line segment RW. Draw a line parallel to line segment QV at point P so that it intersects the parabola diameter at point W' and intersects line EF at F'.

Let the point M is between R and R'. It will be shown $|RM| = |R'M|$.

Consider that line segment QT is on the parabola tangent line at point Q. In accordance with Theorem 6, it is obtained

$$|TV| = 2\,|PV|. \quad (14)$$

Let the line QT intersects line segment EP on point O and line $R'W'$ intersects line segment EQ point F' on. Since line QV parallel to line EP and line PV parallel to line EQ thena $|PV| = |EQ|$ dan $|QV| = |EP|$. Furthermore, from (14), obtained $|TP| = |PV|$. Consequently

$|TP| = |EQ|.$ (15)

As ΔTVQ and ΔTPO are similar and (14), obtained

$|QV| = 2\,|OP|.$ (16)

And

$|EP| = 2\,|OP|.$ (17)

As the result $|EO| = |OP|$.

Based on (16) and (17) it is known that $\Delta\ OEQ$ congruent to ΔOPT. Hence

$|\,\Delta\ QTV| = |EPVQ|.$ (18)

Review ordinate QV and abscissa PV also ordinate $R'W'$ and abscissa PW'. From Theorem 6, obtained

$$\frac{|QV|^2}{|R'W'|^2} = \frac{|PV|}{|PW'|}.$$

Therefore from (18) as well , we get

$|\,\Delta\ R'UW'| = |EPW'F'|.$ (19)

Review ordinate QV and abscissa PV also ordinate RW and abscissa PW. From Theorem 6 obtained

$$\frac{|QV|^2}{|RW|^2} = \frac{PV}{PW}.$$

Therefore from (18) as well, we get

$|\Delta RUW| = |EPWF|.$ (20)

Consider the parallelogram $EPWF$ and $EPW'F'$. The area of the parallelogram $EPWF$ when it subtracted from the area of the parallelogram $EPW'F'$ will result the parallelogram $F'W'WF$. Consider ΔRUW and $\Delta R'UW'$. The area of ΔRUW when subtracted from the area of $R'UW'$ will result polygon $R'W'WFM$. Hence, from (19) dan (20) obtained the area of parallelogram $F'W'WF$ equals to the polygon $R'W'WFM$. Consequently

$$|\Delta\ R'MF'| = |\Delta\ R'MF'|.$$

Since line $F'R'$ is parallel to line FR, then $\angle\ F'R'M = \angle\ FRM$, $\angle\ F'MR' = \angle\ FMR$, and $\angle\ R'F'M = \angle\ RFM$. It means that $\Delta R'MF'$ is congruent to ΔRMF. Thus, $|R'M| = |RM|$. ■

It was found from the proof of Lemma 3 that the remaining area can be treated as parabolic segment having a base from the side of the triangle inscribed earlier and a diameter parallel to the diameter of the parabolic segment whose area is to be determined. As a result, the area comparison of the triangles inscribed on the parabolic segment is obtained as follows.

Theorem 7. *Given parabolic segment QPq. Let V is the center point of the line segment Qq and M is the mid point of the line segment QV. If R is the point of intersection between the parabolic segment with the line through point M and parallel to the diameter of the drawn parabola then,*

$|\Delta QPq| = 8\,|\Delta\{PRQ\}|.$ (21)

Proof : Let point W on the diameter and it is the intersection of a line parallel to line segment QV that passing through point R.

Referring to Theorem 6 it is obtained that $|RW|^2 = |PW| \cdot p$ for a magnitude p parabolic parameter In a similar way we can get that $|QV|^2 = |PV| \cdot p$.

Consequently it obtained

$$\frac{|PV|}{|PW|} = \frac{|QV|^2}{|RW|^2}$$

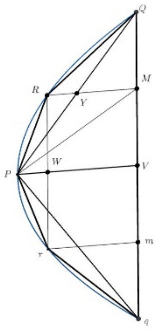

Figure 8. Inscribed triangles of the parabolic segment.

Consequently it obtained

$$\frac{|PV|}{|PW|} = \frac{|QV|^2}{|RW|^2} |PV| = 4\,|PW| = \frac{4}{3}|RM|. \quad (22)$$

Let Y is the intersection of line segments PQ and RM. As RM is parallel to PV and M is the midpoint of QV so that $|PY| = |YQ|$.

Consider that Y is the midpoint PQ, M is the midpoint QV, and ΔPVQ is congruent with ΔYMQ, thus $|PV| = 2|YM|$. Furthermore from (22) obtained,

$$|YM| = 2|RY|. \quad (23)$$

Also consider that ΔPQM and ΔPRQ have the same base, line segment PQ. From (23) and it is known that the angle $\angle RYP = \angle MYQ$, then the height line of ΔPQM is equal to two times with the height line of ΔPRQ. Consequently $|\Delta PQM| = 2\,|\Delta PRQ|$.
Since $|QV| = 2|QM|$, then $|\Delta PQV| = 2\,|\Delta PQM| = 4\,|\Delta PRQ|$ and $|\Delta PQq| = 8\,|\Delta PRQ|$. ■

Since the ratio of the area of inscribed triangles in the n process to the inscribed triangles in the $(n+1)$ process for each $n = 1,2,3, \ldots$, has been known, thus the area of the inscribed triangles in each process can be found out if the inscribed triangles in the $n = 1$ process are known.

Theorem 8. *Given parabolic segment QPq. Let a_1 is the area of ΔQPq. If $a_n = \left(\frac{1}{4}\right)^{n-1} a_1$ for $n =$ 1,2,3, .., then*

$$a_1 + a_2 + a_3 + \ldots + a_N < \text{segment area } QPq$$

For each number N

Proof : Let $a_1 = |\Delta QPq|$. From

Theorem 7 $|\Delta PQq| = 8\,|\Delta PRQ| = 8\,|\Delta Pqr|$ is obtained, then $|\Delta PQq| = 4\,(|\Delta PRQ| + |\Delta Pqr|)$. If $|\Delta PRQ| = a_1$, then

$$|\Delta PRQ| + |\Delta Prq| = \frac{1}{4} a_1 = \frac{1}{4}^{2-1} a_1 = a_2. \quad (24)$$

Based on Lemma 3, if the points S, s, T and t are choosen in accordance with the conditions in Theorem 7, then $|\Delta RSQ| + |\Delta RsP| + |\Delta Ptr| + |\Delta rtq| = \frac{1}{4}|\Delta PRQ| + \frac{1}{4}|\Delta Prq|$. From the equation (24)

$$|\Delta RSQ| + |\Delta RsP| + |\Delta PTr| + |\Delta rtq| = \frac{1}{4}\left(\frac{1}{4}a_1\right) = \frac{1}{4}^{3-1} \quad a_1 \ = \ a_3$$

is obtained.

Furthermore , if the procedure is done repeatedly until the-N process, then the sum of the triangle areas in each process forms the sequence $a_1, a_2, a_3, a_4, \dots, a_N$.

Hence, $a_1 + a_2 + a_3 + a_4 + \dots + a_N$ is the sum of all the triangles' area inscribe the parabolic segment up to the N process. As the inscribing triangle process stop at the-N process, there is a remaining area that is not inscribed by triangle. Consequently $a_1 + a_2 + a_3 + a_4 + \dots + a_N$ is less than the area of the parabolic segment. ■

It has been explained in Theorem 8 that if a parabola is inscribed with triangles in a finite process, the inscribed triangles left space of the parabolic segment. Therefore, knowledge of the infinite series of the area of the inscribed triangles is not enough to determine the area of the parabolic segment. It requires understanding the concept of infinite sum of the triangles inscribing the parabolic segment.

Theorem 9. *Let a_1 is an arbitrary magnitude. If the sequence $a_1, a_2, a_3, \dots$ satisfied $a_n = \frac{1}{4}^{n-1} \ a_1$ for $n = 1,2,3, \dots$, then for every number N holds*

$$a_1 + a_2 + a_3 + \dots + a_N + \frac{1}{3}a_N = \frac{4}{3}\ a_1.$$

Proof : Let

$$b_2 = \frac{1}{3}\ a_2, b_3 = \frac{1}{3}\ a_3, b_4 = \frac{1}{3}a_4, \dots b_N = \frac{1}{3}a_N,$$

Then

$$(a_2 + a_3 + a_4 + \dots + a_N) + (b_2 + b_3 + b_4 + \dots + b_N) = \frac{4}{3}\ (a_2 + a_3 + a_4 + \dots + a_N).$$

Since $a_1 = \ 4a_2, a_2 = 4a_3, a_3 = 4a_4, \dots a_{N-1} = \ 4a_N$, thus,

$$(a_2 + a_3 + a_4 + \dots + a_N) + (b_2 + b_3 + b_4 + \dots + b_N) = \frac{1}{3}\ (a_1 + a_2 + a_3 + a_4 + \dots + a_{N-1}).$$

Yet, sine $(b_2 + b_3 + b_4 + \dots + b_{N-1}) = \frac{1}{3}\ (a_2 + a_3 + a_4 + \dots + a_{N-1})$, then

$$(a_2 + a_3 + a_4 + \dots + a_N) + \ b_N \ = \frac{1}{3}\ a_1. \qquad (25)$$

By adding both segments of (25) with a_1, then

$$(a_1 + a_2 + a_3 + a_4 + \dots + a_N) + \frac{1}{3}a_N \ = \frac{4}{3}a_1. \ ■$$

Theorem 10. *If given a parabolic segment QPq,then its area is equal to four -thirds of the area ΔQPq.*

Proof : Let S is the area of parabolic segment QPq and $K = \frac{4}{3}|\Delta PQq|$. It will be shown $S \ = \ K$. Let $a_1 \ = |\Delta QPq|$, $a_2 = \ |\Delta PRQ| + |\Delta Prq|$, $a_3 = |\Delta RSQ| + |\Delta RsP| + |\Delta PTr| + |\Delta rtq|$, ...Then, the sequence $a_2 = \frac{1}{4}a_1, a_3 = \frac{1}{4}a_2, a_4 = \frac{1}{4}a_3, \dots, a_N = \frac{1}{4}a_{N-1}$ for each number N of the sum of the area of the triangles inscribing the segment in the sequence of the process can be formed. Let $T_N = a_1 + a_2 + a_3 + a_4 + \dots + a_N$.

- Suppose $S < K$.
  Since $a_2 = \frac{1}{4}a_1, a_3 = \frac{1}{4}a_2, a_4 = \frac{1}{4}a_3, \dots$, the sequence $a_1, a_2, a_3, \dots$ satisfies the condition of the Eudoxus principle in Theorem 1. Select $\varepsilon = K - S$. In accordance with the Eudoxus principle, there is a number N such that $a_N < \varepsilon = K - S$. It is then found that
  $$S < T_N - \frac{2}{3}a_N.$$
  for an N. Therefore there is an N that satisfies $S < T_N$. This does not comply with Theorem 8. Thus the assumption of $S \ < \ K$ is incorrect

- Suppose $S > K$
  Consider that $S - T_n < a_n$ for every $n = 1,2,3, \dots$ Choose $\varepsilon = S - K$. Consequently there is a number N such that

$$S - T_N < a_N < \varepsilon = S - K.$$

This implies that $K < T_N$. From Theorem 6 it is found that $T_N + \frac{1}{3}a_N = K$. It means $T_N < K$. This makes the assumption $S > K$ inappropriate.

Thus $S = K$. ■

3.3. Discussion

This article notes that the Eudoxus principle explains the limits of geometric series and in determining the area of parabolic segments by Archimedes using the limits of geometric series. There are no formal concepts of limits and integral concepts in the discussion of these concepts. But through the problem solving, it was realized that the concept of limit was needed in the problem they solved.

The formal concept is not good conveyed too early, but it is introduced by referring to the understanding of concepts that students have acquired in intuition-based contexts previously. This is done as a means of generalization and simplification (Katz, 1998b).

Let ε be a magnitude. As $\varepsilon > 0$, then it has been discovered that the sequence referred to in Eudoxus' principle approaches the number 0 for the-n term toward infinity. The Eudoxus principle can also be used in solving the area of a circle problem; furthermore, it can also be used in solving the area of the shape bounded by a curve problem. The example of solving conveyed by Eudoxus is good to be presented at the beginning of understanding the formal concepts of limits and integral concepts.

A learning approach inspired by history to introduce mathematical concepts or methods can be followed (Gulikers & Blom, 2001). The stages that can be used to introduce mathematical integral concepts or procedures is according to Somaglia (1998). This article recommen three stages to teach the Eudoxus principle to solve a ratio area of two circles.

The teacher should discuss the concept of a circle area which is more familiar to the students in the first stage. Compare the procedure for determining the area of a square and the area of a circle. This stage is conducted at an informal level. The teacher can use colloquial language and graphical representations of a circle and a square whose diameter and diagonal have the same length to stimulate students' intuitive ideas on certain concepts.

Afterwards, the teacher presents the Eudoxus principle (Theorem 1) This theorem conveys the idea of the remaining circle area inscribed by a regular polygon. The Eudoxus principle is very appropriate for preparing students to understand the limit concept. Students have recognized the concept of a sequence and sequence convergence intuitively. They can therefore intuitively understand the sequence convergence referred to by Eudoxus. This stage is to utilize the student's intuition that has been stimulated in the previous stage to outline the main characteristics of the concept.

The last stage is to make a transition to a more formal approach. The formal approach is generally difficult for high school students. In general, they see that the function of proof is just to verify a theorem, while verification is not the point. As a result, students do not understand the meaning of proof. To introduce the mathematical formalization of this concept, teachers provide several appropriate contexts in which students can work informally on activities related to proofs. The exercise leads the students to prove that the sequence of remainders of a circular region inscribed by a regular polygon is as follows.

1. Take any magnitude of the radius a circle $\mathcal{C}$. Let M_0 is the area of $\mathcal{C}$.
2. Inscribe $\mathcal{C}$ with a square $\mathcal{P}_1$ with the area $a(\mathcal{P}_1)$. Let's find $M_1 = M_0 - a(\mathcal{P}_1)$!
3. Compare M_1 and M_0. Which one is greater?
4. Inscribe each remaining area with an isosceles triangle. If $a(\mathcal{P}_2)$ is the sum of $a(\mathcal{P}_1)$ and the isosceles triangles, find $M_2 = M_0 - a(\mathcal{P}_2)$ and compare M_2 and M_1. Which one is greater?
5. Repeat the procedure of point 5 for each remaining area and find $M_k = M_0 - a(\mathcal{P}_k)$ for $k = 3,4,5,\ldots$
6. Do the magnitudes M_1, M_2, M_3, M_4.. satisfy the Eudoxus principle? Why?
7. Take any magnitude $\varepsilon < M_0$. Is there a remaining area where area M_k satisfy $M_k < \varepsilon$?

Mathematical proof can have several functions, including verification, explanation, systematization, discovery and communication (Rocha, 2019). In formal proof, there is no activities such as using an example of magnitude or calculating the magnitudes. Since the exercise is to transition the informal proof to formal proof, the explanation function is the most relevant one to deserves special attention. So, it is not a problem if the students still should use a particular magnitude to verify their statements. However, it cannot be denied that the teacher's guidance about the quantifier is still needed.

To prove that the limit concept is used for the integral problem (Theorem 3), the following exercise can be used.

1. Take 2 arbitrary circle $\mathcal{C}_1$ and $\mathcal{C}_2$ with radius r_1 and r_2.
2. Let S be the area win hich the area magnitude is $S = \frac{r_2}{r_1} a(\mathcal{C}_1)$. Find S!
3. Imagine S is less than $a(\mathcal{C}_2)$. Take any ε which satisfies $0 < \varepsilon < a(\mathcal{C}_2) - S$? Why it can be done?
4. Inscribe $\mathcal{C}_2$ with a regular polygon $\mathcal{P}_2$. Can you choose $\mathcal{P}_2$ with the area $a(\mathcal{P}_2)$ which satisfy $a(\mathcal{C}_2) - a(\mathcal{P}_2) < \varepsilon$? Explain your answer.
5. If you could create $\mathcal{P}_2$ that satisfy $a(\mathcal{C}_2) - a(\mathcal{P}_2) < \varepsilon$. Compare $a(\mathcal{P}_2)$ and $a(\mathcal{C}_2)$ (please look again to point 3).
6. Note that $a(\mathcal{C}_1) > a(\mathcal{P}_1)$ with $\mathcal{P}_1$ is a regular polygon with the radius is r_1 and has the number of sides is the same as $\mathcal{P}_2$. Recall that $\frac{a(\mathcal{P}_1)}{a(\mathcal{P}_2)} = \frac{r_1^2}{r_2^2}$. If $S = \frac{r_2}{r_1} a(\mathcal{C}_2)$, is the magnitude of S is less than $a(\mathcal{C}_2)$?
7. Compare your work on point 3 and 6. What can you conclude?

Next, the teacher also provides similar exercises but with condition S is greater than $a(\mathcal{C}_2)$ to other students. With this way of exercising, students will be engaged in tasks in which they can experiment with different proof functions.

Archimedes' problem solving of parabolic segment area also did not use the integral concept that had been discovered and was part of the material learned in high school. Rather than choosing a square or trapezoid to inscribe the parabolic segment, Archimedes prefers to use an inscribed triangle to approximate the area of the parabolic segment. This also motivates students to think creatively. Furthermore, students' intuitive understanding of the monotonous descending series formed from the magnitude of the areas of the inscribed triangles can also direct students that the infinite series referred to by Archimedes will lead to a certain value.

Since the parabola figure is not familiar to students. The teacher could use a graphical representations of a parabolic segment in which the magnitude length of diameter and parabola base are given to stimulate pupils' intuitive ideas on certain concepts. Afterward, the teacher presents Theorem 4. This theorem conveys the idea of the ratio of the inscribing area by triangles. This is very appropriate to deliver to students to understand how the area of the parabola segment can be solved by summing the inscribing triangle. Students have recognized the concept of partial summation concept. This stage is to exploit the student's intuition that has been stimulated in the previous stage to outline the main characteristics of the series concept. The teacher should guide the students on how to construct the diameter or an arbitrary parabola segment and the inscribing triangle based on Definition 11 and Lemma 3.

To prove Theorem 10, the following exercise can be used.

1. Create an arbitrary parabola segment figure with the diameter line.
2. Choose a number N. Construct N inscribing some triangles with area $a_1, a_2, a_3, \dots a_N$. Calculate $T_N = a_1 + a_2 + a_3 + \dots + a_N$.
3. Is the area of the parabola segment (S) greater than the inscribing triangles constructed? Why?
4. Calculate $K = a_1 + a_2 + a_3 + \dots + a_N + \frac{1}{3} a_N$. What is the ratio K to the a_1?
5. Choose any $\varepsilon > a_N$. Suppose $S = K$- ε. It means $S < K$. Is T_N greater than S? Explain!
6. Does the result on point 5 comply with point 3?
7. Choose any $\varepsilon > a_N$. Suppose $S = K + \varepsilon$. It means $S > K$. Is T_N greater than K? Clarify the comparison with point 2 and 4 results.

The exercise should be done together with the teacher guidance. Especially when students do the contradiction proof.

The proof is done without a formal limit definition. This will bridge students' thinking on the importance of limit and integral concepts to be used generally if the cases encountered are different.

The explanation level in this article is slightly higher than those done at the high school. To be more easily understood, the reccomendation for the teachers are :

1. Emphasize to students the background of the problem-solving done by Eudoxus and Archimedes;
2. The limit existence discussion is not done first in the delivery of the problem solving. Instead, it was explained by the teacher after the students intuitively understand the case delivered;

3. It is recommended that teachers provide another case similar to the problems discussed in this article after explaining it;
4. The teacher can also deliver about the π number related to the circle problem which is done by Eudoxus.

4. Conclusion

Eudoxus' principle played a major role in the discovery of the circle area formula. Even though the modern Eudoxus' principle can be generalized using the concept of limit, it is the simple idea of Eudoxus' principle that gives students an understanding of limit that helps the integral formulation

Besides the Eudoxus principle, the exhaustion method used by Archimedes also succeeded in providing an alternate calculation of parabolic segment area. The propositions made and proof by Eudoxus and Archimedes are simpler than the formal definition of limit in mathematics. This bridges the previous knowledge that students already have to understand the formal definition of limits.

The cases of solving passed on by Eudoxus and Archimedes are good to be presented at the starting of understanding the formal concepts of limits and integral concepts. This article provide three stages on how to deliver the content based on somaglia's practical example. In the first stage, the teacher should use circle and parabola segment representation which students are familiar with. Next, the teacher stimulates pupils' intuitive ideas by presenting the core theorem to the students. The theorem should be delivered so students would get the prior understanding about Eudoxus' problem-solving are Theorem 1 and Theorem 3. Theorem 10 is needed to stimulate the students' ideas about Archimedes' problem-solving. The last step is transitioning students' intuitive ideas to a more formal approach using the proof. To introduce the mathematical formalization of this concept, the teachers should use the exercise that directs the students to work informally on activities related to proofs.

The solution process carried out by Eudoxus and Archimedes not only involved calculating procedures, but also involved understanding concepts. Using the technical knowledge provided by Archimedes before being introduced to the concept of integral, students are expected to get a good connection between Archimedes' method and the modern basic concept of integral.

References

Ball, W. W. R. (1960). *A short account of the history of mathematics*. Dover publ.

Bauldry, W. C. (2009b). *Introduction to real analysis: An educational approach*. Wiley.

Boyer, C. B. (1988). *The history of the calculus and its conceptual development: The concepts of the calculus* (Repr). Dover Publ.

Capaian Pembelajaran pada Pendidikan Anak Usia Dini, Jenjang Pendidikan Dasar, dan Jenjang Pendidikan Menengah pada Kurikulum Merdeka. (n.d.). Kementrian Pendidikan dan Kebudayaan Indonesia. https://kurikulum.kemdikbud.go.id/wp-content/unduhan/CP_2022.pdf

Donovan, S., Bransford, J., & National Research Council (U.S.) (Eds.). (2005). *How students learn: Mathematics in the classroom*. National Academies Press.

Furinghetti, F., & Somaglia, A. (1998). History of Mathematics in School across Disciplines. *Mathematics in School*, *27*(4), 48–51. JSTOR.

Gulikers, I., & Blom, K. (2001). A historical angle', a survey of recent literature on the use and value of history in geometrical education. *Educational Studies in Mathematics*, *47*, 223–258.

Harris, D. J. (2020). *Literature review and research design: A guide to effective research practice*. Routledge, Taylor & Francis Group.

Katz, V. J. (1998b). *A history of mathematics: An introduction* (2nd ed). Addison-Wesley.

Kaygin, B., Balçin, B., Yildiz, C., & Arslan, S. (2011b). The effect of teaching the subject of Fibonacci numbers and golden ratio through the history of mathematics. *Procedia - Social and Behavioral Sciences*, *15*, 961–965. https://doi.org/10.1016/j.sbspro.2011.03.221

Kilpatrick, J., Swafford, J., Findell, B., & National Research Council (U.S.) (Eds.). (2001). *Adding it up: Helping children learn mathematics*. National Academy Press.

Paudel, H. P., Alt, C., Runnels, J., & Lin, C. P. (2018). Pupil plane differential detection microscopy. *Optics Letters*, *43*(18), 4410–4412. https://doi.org/10.1364/OL.43.004410

Rocha, H. (2019). Mathematical proof: From mathematics to school mathematics. *Philosophical Transactions of the Royal Society A*, *377*(2140), 20180045.

Sjamsuddin, H. (2016). *Metodologi sejarah*. Penerbit Ombak.

Sukarani, N. M., & Bella, C. (2022). Sejarah Aritmatika: Manfaat Pembelajaran Sejarah Matematika. *Jurnal Dunia Ilmu*.

Susilo, B. E., Darhim, D., & Prabawanto, S. (2019b). Kesulitan Belajar Mahasiswa pada Materi Aplikasi Integral untuk Luas Daerah dalam Perspektif Disposisi Matematis. *Kreano, Jurnal Matematika Kreatif-Inovatif*, *10*(1), 86–93. https://doi.org/10.15294/kreano.v10i1.19373

Topolski, J. (1977). *Methodology of History*. Springer Netherlands.

Utami, W. B., Widodo, S. A., & Sulistyowati, F. (2022). *Buku Panduan Guru Matematika Tingkat Lanjut untuk SMA/MA Kelas XII*. Pusat Perbukuan Indonesia. https://buku.kemdikbud.go.id/katalog/buku-panduan-guru-matematika-tingkat-lanjut-untuk-smama-kelas-xii