SOLUTION OF HARMONIC OSCILLATOR OF NONLINEAR MASTER SCHRÖDINGER

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Abstract

We have computed the solution of a nonrelativistic particle motion in a harmonic oscillator potential of the nonlinear master Schrödinger equation. The equation itself is based on two classical conservation laws, the Hamilton-Jacobi and the continuity equations. Those two equations give each contribution for the definition of quantum particle. We also prove that the solution can't be normalized.

Keywords : harmonic oscillator, nonlinear Schrödinger.

1. Introduction

In the twentieth century, there were some interesting discussions about quantum theory viewed both mathematical and physical aspects. In this time, we only consider about two cases. One of them, the recent paper [1] which showed that the Schrödinger equation could be derived by Newtonian mechanics which is based on the assumption that every particle of mass m follows the Brownian motion with no friction and diffusion coefficient $\hbar/2m$. In the other side, Feynman could formulate the derivation of Schrödinger equation from path integral concept which has been taken from Dirac's idea that all classical paths contribute to transition amplitude; for further reading see [2]. It is clear that Schrödinger equation can be derived by any other interpretations of the particle motion.

The second one is about oscillator harmonic model. The harmonic oscillator has been discussed in many quantum areas and its model has been applied to explain the microscopic phenomena [3]. All the models of harmonic oscillator have each unique interpretation. In the recent paper [4], the relativistic oscillator model, in case of the generalized Schrödinger picture, gives us the information that the operators of spacetime independent in that state induce the spacetime dependent operators related to the Killing vectors of the AdS space. Beside of that, the ground state energy of the model can not define like the ground state energy for nonrelativistic harmonic oscillator in quantum mechanics. In the other paper [5], the mechanical system at the horizon in AdS_2 can be considered as thermal harmonic oscillator which its temperature will be given by Hawking temperature at the horizon.

In this paper, we consider another model of Schrödinger equation, named nonlinear master Schrödinger. The main purpose of this paper itself is to mathematically discuss the nonrelativistic motion of particle in harmonic oscillator potential for the nonlinear master Schrödinger equation which was initially proposed by Guerra, Pusterla, and Smolin in order to build the general theory of quantum mechanics. The detail explanations of nonlinear master Schrödinger can be found in the references [6-9]. The main reason of the equation's construction was inspired from Einstein and de Broglie who believe that the general theory of quantum should be nonlinear. The formulation itself did not include two fundamental postulates in ordinary quantum theory, but it was built from Hamilton-Jacobi and continuity equations. However, we also realize that the superposition principle generally does not hold anymore. In addition, the general solution which has been proposed will not generally give us the normalization constant because of the spacetime dependent amplitude.

For the short review, according to them, a quantum particle, at the same time has two parts, a localized wave (singularity part) and a cloud of particle (extended part). Hamilton-Jacobi describes the particle motion which is called as singularity part; otherwise the continuity equation describes the wave motion which is called the extended part. It is clear that the singularity part has the role as the nucleus which has almost a quantum particle energy which is concentrated; otherwise the extended part has the role as the cloud which surrounds the singularity part. In addition, we can conclude that the singularity part is responsible for the detection in particle detector and the extended part is responsible for interference process. For the general case, we have difficulties to make the mathematical function model describing the singularity and extended parts. In reviewed articles [10, 11], the realistic model can be formulated only for the free particle solution. For this solution, the amplitude has the linear differential equation, so the superposition principle must be hold.

In the language of nonlinear master Schrödinger, unfortunately, we can't make associate with eigenvalue such as in ordinary quantum mechanics. However, one of the interesting cases is that the solution can have the solution like soliton. From many textbooks, we know that the soliton solution becomes a favorite model to describe the particle rather than wave packets because of its characteristic solution. This case was initially studied by Gueret and Vigier by adding a new parameter in quantum potential term [10].

2. Mathematical Solution

First, we initially consider about the mathematical formula about the nonlinear master Schrödinger which combine classical Hamilton-Jacobi and continuity equations, for further details can be found in [6, 9] or for reviewing see also [10]. In one-dimensional case the corresponding equation is expressed as

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+\frac{\hbar^2}{2m}\frac{1}{a}\frac{\partial^2 a}{\partial x^2}\psi+V\psi=i\hbar\frac{\partial\psi}{\partial t}, \quad (1)$$

with the general solution

$$\psi(x,t)=a(x,t)\ e^{\frac{i}{\hbar}\varphi(x,t)}, \quad (2)$$

and the term

$$Q=\frac{\hbar^2}{2m}\frac{1}{a}\frac{\partial^2 a}{\partial x^2} \quad (3)$$

is called the quantum potential. Then, the corresponding Hamilton-Jacobi and the continuity equations for the harmonic oscillator potential can be written

$$\frac{1}{2m}\left(\frac{\partial\varphi}{\partial x}\right)^2+\frac{1}{2}m\omega^2x^2=-\frac{\partial\varphi}{\partial t}, \quad (4)$$

$$\frac{1}{m}\frac{\partial}{\partial x}\left(a^2\frac{\partial\varphi}{\partial x}\right)=-\frac{\partial a^2}{\partial t}. \quad (5)$$

The formulation of the corresponding Hamilton-Jacobi and continuity equations are given in appendix.

We make an ansatz solution for (4) as the separation of variables between space and time

$$\varphi(x,t)=\varphi_1(t)+\varphi_2(x). \quad (6)$$

It is easy to show that the final solution for the phase has the form

$$\varphi(x,t)=-Et+m\omega\left\{\frac{x}{2}\sqrt{\frac{2E}{m\omega^2}-x^2}+\frac{E}{m\omega^2}\sin^{-1}\left(x\sqrt{\frac{m\omega^2}{2E}}\right)\right\}, \quad (7)$$

where E is total energy of the system.

The amplitude solution can be obtained by substitution (7) into (5) and using the separation of variables again for the amplitude as

$$a(x,t)=\mathrm{X}(x)T(t),$$

(8)

one will get the solution

$$a(x,t)=K\frac{\exp\left\{\frac{iC}{2\hbar}\left[\sin^{-1}\left(x\sqrt{\frac{m\omega^2}{2E}}\right)-\frac{t}{m}\right]\right\}}{\left(2mE-m^2\omega^2x^2\right)^{\frac{1}{4}}}, \quad (9)$$

where K and C are constants. Here we have inserted the constant $i/\hbar$ in the solution for the time part in order to get the normalization constant, so we expect to get the value of probability to find the particle in the certain interval space.

The wave solution for the nonrelativistic particle motion in a harmonic oscillator potential for the nonlinear master Schrödinger equation can be written as

$$\psi(x,t)=K\frac{\exp\left\{\frac{iC}{2\hbar}\left[\sin^{-1}\left(x\sqrt{\frac{m\omega^2}{2E}}\right)-\frac{t}{m}\right]\right\}}{\left(2mE-m^2\omega^2x^2\right)^{\frac{1}{4}}}\exp\left\{\frac{i}{\hbar}\left(-Et+m\omega\left[\frac{x}{2}\sqrt{\frac{2E}{m\omega^2}-x^2}+\frac{E}{m\omega^2}\sin^{-1}\left(x\sqrt{\frac{m\omega^2}{2E}}\right)\right]\right)\right\}. \quad (10)$$

We can see that the wave function can't be normalized for all space.

3. Conclusion

In this paper we have mathematically considered the nonrelativistic particle motion in harmonic oscillator potential for nonlinear master Schrödinger. In using the nonlinear master Schrödinger, one has to assume that two classical motion equations, Hamilton-Jacobi and continuity equations, still hold in microscopic level. For the solution of the harmonic potential, we found that the wave function can't be normalized rather than in ordinary quantum mechanics at the same potential.

The final result of the wave function has shown to us that no corresponding eigenenergy in this system. In addition, we also can not calculate the observable quantities like in ordinary quantum mechanics, such as the expectation of energy, momentum, and even position. However, we still have chance to define the normalization constant, so the probability finding the particle for the certain interval in the space can be found. From the solution above, we also see that the wave solution can't define the singularity and extended parts.

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5. Appendix

The nonlinear master Schrödinger equation in three dimensions has the form

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \frac{\hbar^2}{2m}\frac{\nabla^2 a}{a}\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}, \qquad (11)$$

with the general solution

$$\psi(\vec{x},t) = a(\vec{x},t)\, e^{\frac{i}{\hbar}\varphi(\vec{x},t)}. \qquad (12)$$

In the classical mechanics, three-dimensional Hamilton-Jacobi and continuity equations are defined

$$\frac{1}{2m}(\nabla S)^2 + V = -\frac{\partial S}{\partial t}, \qquad (13)$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0, \qquad (14)$$

where all the symbols have their usual meaning. For more detail, one can see the following textbooks [12, 13]. In the nonlinear master Schrödinger, those symbols are transformed by the following relations

$$S = \varphi,\ \vec{p} = \nabla\varphi,\ E = -\frac{\partial\varphi}{\partial t},$$

$$\vec{v} = \frac{\vec{p}}{m},\ \rho = a^2,\ \vec{J} = \rho\,\vec{v}, \qquad (15)$$

and the normalization condition for all space reads

$$\int \rho(\vec{x},t)\, d^3x = 1. \qquad (16)$$

For finding the solution for amplitude and phase in the equation above, one can use the result of the following integral

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \qquad (17)$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right). \qquad (18)$$

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