Sequential Detection under Correlated Observations using Recursive Method

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Abstract— Sequential analysis has been used in many cases when the decision is required to be made quickly, such as for signal detection in statistical signal processing, namely sequential detector. For identical error probabilities, a sequential detector needs a smaller average sample number (ASN) than its counterpart of a fixed sample number quadrature detector based on Neyman-Pearson criteria. The optimum sequential detector was derived based on the assumption that the observations are uncorrelated (independent). However, the assumption is commonly violated in realistic scenario, such as in radar. Using a sequential detector under correlated observations is sub-optimal and it poses a problem. It demands a high computational complexity since it needs to recalculate the inverse and the determinant of the signal covariance matrix for each new sample taken. This paper presents a technique for reducing computational complexity, which involves using recursive matrix inverse to calculate conditional probability density functions (pdf). This eliminates the need to recalculate the inverse and determinant, leading to a more reasonable solution in real-world scenarios. We evaluate the performance of the proposed (recursive) sequential detector using Monte-Carlo simulations and we use the conventional and non-recursive sequential detectors for comparisons. The results show that the recursive sequential detector has equal probabilities of false alarm and miss-detection with the conventional sequential detector and performs better than the non-recursive sequential detector. In terms of ASN, it maintains results comparable to those of the two conventional detectors. The recursive approach has reduced the computational complexity for matrix multiplication to $\mathcal{O}(\mathrm{n}^2)$ from $\mathcal{O}(\mathrm{n}^3)$ and has rendered the calculation of matrix determinants unnecessary. Therefore, by having a better probability of error and reduced computational complexities under correlated observations, the proposed recursive sequential detector may become a viable alternative to obtain a more agile detection system as required in future applications, such as radar and cognitive radio.

Keywords— Covariance matrix; matrix determinant; matrix inverse; Neyman-Pearson; quadrature detector; sequential analysis; sequential detector

I. Introduction

The introduction of sequential analysis as an alternative method in hypothesis testing by Abraham Wald in 1945 [1] has drawn the attention from many researchers to extend it to the fields that need fast decision making (small average sample numbers). One of the frequently used sequential analyses for hypothesis testing is the Sequential Probability Ratio Test (SPRT) [1], which was proven to be optimum in terms of the average sample number (ASN) compared to other possible sequential schemes. Since then, SPRT has not only been of interest to statisticians but also to researchers in engineering. In statistical signal processing, signal detection based on SPRT, or sequential detector, has been intensively researched and developed. The first classic reference to sequential detector can be found in [2], which is then followed by some classical works to solve the problem of possibly having huge sample numbers [3], [4] and to derive its asymptotic efficiency [5], [6]. Recently, sequential detectors have been applied not only using a single sensor node but using many in a centralized or decentralized distributed manner [7]-[9], to improve performance. This happens along with the widespread use of wireless sensor networks to detect physical phenomena in a geographical area [10]. A tutorial on sequential detection that discusses classical theoretical results and recent developments emphasizing Quickest Change Detection can be found in [11] and the references therein.

In contrast to the Neyman-Pearson detector, which only has one threshold [12], a sequential detector introduces two thresholds at the detector output so that the signal is reported present if one is exceeded and absent if the other is exceeded. The length of the detection process (sample number) is not predetermined but rather a random variable that varies with the course of the test. For the same target error probability, a sequential detector has the advantage of having a smaller ASN than its counterpart Neyman-Pearson detector with a fixed sample number. Consequently, a sequential detector is more agile than a Neyman-Pearson detector. Due to its agility, the sequential detector is later considered as an appropriate solution for target detection in radar [13] and for spectrum sensing in cognitive radio [14].

Some issues on optimal quantization for sequential detectors have been addressed in [15]. Sequential detectors can also be used to detect the presence of anomalies in networks with parallel data streams or multiple data sources, including when sampling constraints exists [16], [17], both

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parametrically [18] and non-parametrically [19]. Implementations of sequential detectors could also be found in multisensory integrated systems [20] and the Internet of Things (IoT) [21].

Sequential analysis has been broadly used not only in electrical engineering but also in materials engineering [22], nuclear engineering [23], and reliability testing [24]. In medicine, sequential analysis is applied for clinical trials to find the best treatment for Covid-19 patients [25]. It has been used in environmental issues to detect changes in carbon dioxide emission levels [26]. Two sequential methods, SPRT and Sequential Bayes Factor Test (SBFT), are recommended in psychological research [27]. Even in politics, sequential analysis has been used, e.g., to generate evidence in determining the level of decline in trust in the government [28].

A critical issue to be addressed in signal detection using a sequential detector is the degree of correlation between observations (correlated or uncorrelated) [7] since it might significantly affect the performance if not carefully examined. The optimum sequential detector was derived, assuming that the observations are uncorrelated (independent). However, the assumption is frequently not met in practice, as seen in radar and cognitive radio. Using a sequential detector under correlated observations presents an issue with computational complexity [29]. In the detection process, a sequential detector recalculates the inverse and determinant of the signal covariance matrix each time a new sample is received. The computational complexity increases since each calculation must be delivered as the size of the signal covariance matrix expands to accommodate the new sample. Therefore, this complexity should be addressed rigorously.

Unlike most of the literature cited above, this paper aims to reduce the computational complexity of a sequential detector under correlated observations, making it more feasible in practice. The method proposed in this paper proves that recursively calculating the matrix inverse and matrix determinant can reduce the computational complexity without affecting the performance of the error probability and the average sample number (remaining the same as the conventional sequential detector). By doing so, an agile detection system is more viable, as required in radar and cognitive radio. To the best of our knowledge, this is rarely discussed in the literature. Thus, the contributions of the paper are:

- A recursive implementation of sequential detection has been successfully derived, whereas the previous literature mostly discusses non-recursive methods.
- An alternative agile sequential detector with less computational complexity can be obtained, under correlated observations, making it more feasible for use in a in real scenario.

To make the paper easy to follow, next, it is organized into 4 sections. The signal model is presented in Section II. It also discusses two conventional approaches for signal detection when the observations are correlated, namely the quadrature detector and the conventional sequential detector, followed by the proposed method of recursive sequential detector. Section II ends with the simulation scenario. The simulation results and computational complexities of recursive, non-recursive, and conventional sequential detectors are discussed in Section III. Concluding remarks can be found in section IV.

II. Method

In this section, we begin with a brief description of the signal model, with the aim of contextualizing signal detection as a binary hypothesis testing problem, i.e. signal absent versus signal present. This is followed by a description of two conventional detectors for binary hypothesis testing problems, namely the Neyman-Pearson-based detector with a fixed sample number (quadrature detector) and the conventional sequential detector with random samples. This section closes with the formulation of the proposed solution, the recursive sequential detector, which minimizes the sequential detector's computational complexity while dealing with correlated observations. We use simulation to evaluate the performance of the proposed method and compare it to the conventional methods. The performance metrics that we use are probability of false alarm, probability of miss detection and average sample number.

A. Signal Model and Conventional Approaches

Suppose that a receiver using a sequential detector observes a sequence of random signal samples $x_1, x_2, \ldots$ up to the nth sample, and be represented by a signal vector $\mathbf{x}_n$. The problem of signal detection can be formalized as a binary hypothesis testing between the null hypothesis ($\mathcal{H}_0$) representing that the signal is absent (noise only) and the alternative hypothesis ($\mathcal{H}_1$) representing that the signal is present (signal plus noise) [12],

$$\begin{aligned} \mathcal{H}_0 &: \mathbf{x}_n = \mathbf{w}_n \\ \mathcal{H}_1 &: \mathbf{x}_n = \mathbf{r}_n + \mathbf{w}_n, \qquad n = 1,2, \ldots. \end{aligned} \tag{1}$$

Here, $\mathbf{r}_n \in \mathbb{C}^n$ is an n dimensional signal vector having correlated complex Gaussian distribution with zero mean and covariance matrix $\mathbf{R}_{rn} = E[\mathbf{r}_n \mathbf{r}_n^H]$. $(\cdot)^H$ and $E[\cdot]$ denote Hermitian transpose and expectation, respectively. $\mathbf{w}_n$ represents a noise vector having identically independent distributed (i.i.d) Gaussian with covariance matrix $\mathbf{R}_{wn} = \sigma^2 \mathbf{I}$ with $\mathbf{I}$ represents identity matrix. Thus, (1) can be rewritten as

$$\begin{aligned} \mathcal{H}_0 &: \mathbf{x}_n \sim \mathcal{CN}(\mathbf{0}, \sigma^2 \mathbf{I}) \\ \mathcal{H}_1 &: \mathbf{x}_n \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}_n + \sigma^2 \mathbf{I}). \end{aligned} \tag{2}$$

Here $\sim\mathcal{CN}(\boldsymbol{\mu}, \mathbf{R})$ denotes distributed according to Gaussian complex with mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{R}$. The covariance matrix $\mathbf{R}_n = \mathbf{R}_{rn} + \sigma^2 \mathbf{I}$ is Hermitian and Toeplitz for each n, and the signal is assumed to be stationer.

Two conventional approaches in signal detection under correlated observations are quadrature detector based on the Neyman-Pearson criterion with a fixed sample number and sequential detector with random sample numbers. A brief description of both detectors will be provided in the next sub-sections to gain a thorough understanding of the problem being addressed in this paper. In addition, these two detectors will serve as references to compare to when we evaluate the performance of the proposed sequential detector.

1) The Quadrature Detector

The quadrature detector is derived based on the Neyman-Pearson criterion, which is to maximize the probability of detection P_d with the pre-specified probability of false alarm P_f. This approach leads to the Likelihood Ratio Test (LRT) with a fixed sample number [12]. Suppose that N is the sample number, the joint probability density function of the vector $\mathbf{x}_N$ under $\mathcal{H}_0$ can be written as

$$\mathrm{p}_0(\mathbf{x}_N) = \frac{1}{\pi^N \sigma^{2N}} \exp\left(-\frac{1}{\sigma^2} \mathbf{x}_N^H \mathbf{x}_N\right), \tag{3}$$

and under $\mathcal{H}_1$ as

$$\mathrm{p}_1(\mathbf{x}_N) = \frac{1}{\pi^N \det(\mathbf{R}_N)} \exp\left(-\frac{1}{\sigma^2}\mathbf{x}_N^H \mathbf{R}_N^{-1}\mathbf{x}_N\right). \quad (4)$$

Where $(\cdot)^{-1}$ and $\det(\cdot)$ denote matrix inverse and matrix determinant, respectively. Using (3) and (4), the LRT ($\mathcal{L}_N$) is

$$\begin{aligned}\mathcal{L}_N &= \frac{\mathrm{p}_1(\mathbf{x}_N)}{\mathrm{p}_0(\mathbf{x}_N)} \\ &= \frac{\sigma^{2N}}{\det(\mathbf{R}_N)} \exp\left(-\mathbf{x}_N^H\left[\mathbf{R}_N^{-1} - \frac{\mathbf{I}}{\sigma^2}\right]\mathbf{x}_N\right).\end{aligned} \quad (5)$$

After some manipulations, the quadrature detector Q_N can be expressed by [12],

$$Q_N = \mathbf{x}_N^H[\mathbf{R}_{rN}\mathbf{R}_N^{-1}]\mathbf{x}_N \begin{cases} \geq \tau, & \text{accept } \mathcal{H}_1 \text{ (signal present)} \\ < \tau, & \text{accept } \mathcal{H}_0 \text{ (signal absent)} \end{cases}, \quad (6)$$

where τ is the pre-specified threshold based on the target P_f.

2) The Conventional Sequential Detector

The second approach is a conventional sequential detector based on SPRT [1]. The advantage of SPRT is that the two error probabilities, the probability of miss detection P_m and the probability of false alarm P_f, can be kept constant at a certain level following the requirement. However, this results in random sample numbers.

Unlike the quadrature detector, the SPRT uses two constants for the thresholds, i.e., an upper threshold A to control P_f dan a lower threshold B to control P_m. The detector will stop sampling and then decide if either threshold A or B is crossed for the first time. Otherwise, the detector will continue to take a new sample ($n \leftarrow n+1$). The stopping time N_s of a sequential detector can be defined as

$$N_s = \min_{n\geq 1}\{n: \mathcal{L}_n \notin (B, A)\}, \quad (7)$$

where $\mathcal{L}_n$ is the LRT until the nth sample. The test will decide $\mathcal{H}_1$(signal is present) if $\mathcal{L}_{N_s} \geq A$, and $\mathcal{H}_0$ (signal is absent) if $\mathcal{L}_{N_s} < A$. Thus, the sequential detector reads as follow,

$$\mathcal{L}_n = \frac{\mathrm{p}_1(\mathbf{x}_n)}{\mathrm{p}_0(\mathbf{x}_n)} \begin{cases} \geq A, & \text{accept } \mathcal{H}_1 \\ < B, & \text{accept } \mathcal{H}_0 \\ A < \mathcal{L}_n < B, & n \leftarrow n+1 \text{ (take a new sample).} \end{cases} \quad (8)$$

The two thresholds A and B can be found using Wald's approximation [1], i.e.,

$$A = \frac{1-\beta}{\alpha}, \quad \text{dan} \quad B = \frac{\beta}{1-\alpha} \quad (9)$$

where α is the nominal probability of false alarm and β is the nominal probability of miss detection.

If P_f and P_m represent the actual probabilities of false alarm dan miss-detection, respectively, the following two inequalities are guaranteed [2],

$$P_f \leq \frac{\alpha}{1-\beta}, \quad \text{and} \quad P_m \leq \frac{\beta}{1-\alpha}. \quad (10)$$

Nominal α and β will be determined by requirement and usually pre-specified to be very small, so as to have $P_f \sim \alpha/(1-\beta)$ and $P_m \sim \beta/(1-\alpha)$. Thus, the probability of having the actual P_f larger than the nominal α and having the actual P_m larger than the nominal β will be very small and can be neglected in the application.

For the detection model (2) and the LRT (5), after using logarithmic function, the test statistic for the conventional sequential detector (8) becomes,

$$S_n = \mathbf{x}_n^H\left[\frac{\mathbf{R}_{rn}\mathbf{R}_n^{-1}}{\sigma^2}\right]\mathbf{x}_n + [n\log(\sigma^2) - \log\det(\mathbf{R}_n)] \begin{cases} \geq \log(A), & \text{accept } \mathcal{H}_1 \\ < \log(B), & \text{accept } \mathcal{H}_0 \\ \log(A) < S_n < \log(B), & n \leftarrow n+1. \end{cases} \quad (11)$$

Now, the conventional sequential detector based on the SPRT has been completely defined.

In sequential detector, the sample number is random depending on statistics of the received signal. The higher the sample number, the larger the signal covariance matrix. This will create problems when calculating the inverse and determinant of the covariance matrix $\mathbf{R}_n$ in (11). For example, when the Signal to Noise Ratio (SNR) is small, the detector needs a very large sample number n (or large stopping time N_s) because it needs more information to distinguish between two similar states, i.e., a state of no signal (noise only) and another state of signal drowned in noise due to small SNR. Thus, in this case, the covariance matrix $\mathbf{R}_n$ continues to grow very large and so the computational complexity for recalculating its inverse and determinant becoming very high as the sample number n increases. This is not the case for the quadrature detector with the fixed sample number, since the inverse and determinant only need to be calculated once to decide.

Up to this point, it has been demonstrated that using sequential detectors under correlated observations will result in significant computational complexity. In the following part, we will describe how to solve this complexity by using recursive matrix inversion, which is then used to construct the probability density function under the alternative hypothesis ($\mathcal{H}_1$).

B. Recursive Sequential Detector (Proposed Method)

A solution for the implementation of sequential detector under correlated observation could be to calculate the inverse and determinant of matrix $\mathbf{R}_n$ in advance and store it in a memory. However, there are three reasons why this is not the case. First, this study assumes that truncation such as in [30], [31] is not conducted, so the maximum sample number $\max(N_s)$ is not known in advance. Secondly, the amount of memory space M to store the inverse matrices will depend on the random $\max(N_s)$. Thus, even in a perfect scenario, the required storage can easily exceed the memory size provided by small devices, such as sensor nodes or hand-held devices. Thirdly, the main target of this method is blind detection with unknown parameters (the covariance matrix in this case), so the matrix calculation must be delivered real time with the smallest possible computational complexity. Thus, to derive an efficient method for implementing sequential detector under correlated observation is necessary.

One way to solve the computational complexity of sequential detector under correlated observations is to approximate $\mathbf{R}_n$ in (11) by its corresponding circular matrix $\mathbf{C}_n$. This method has been proposed in [29], namely non-recursive sequential detector. This method will also serve as a reference to evaluate the proposed sequential method (see Results and Discussion). Instead of using approximation, the proposed method is primarily based on facilitating the calculation of the inverse of the current covariance matrix $\mathbf{R}_n^{-1}$ based on the knowledge of the inverse of the previous covariance matrix$\mathbf{R}_{n-1}^{-1}$ (recursively). Therefore, from this point forward, the method will be refered to as recursive sequential detector.

As the first step, (5) will be extended using the chain rule of probability. So, the LRT can be expressed as conditional probabilities,

$$\mathcal{L}_n = \frac{\mathrm{p}_1(\mathbf{x}_n)}{\mathrm{p}_0(\mathbf{x}_n)} = \prod_{i=1}^{n} \frac{p_1(x_i|\mathbf{x}_{i-1})}{p_0(x_i|\mathbf{x}_{i-1})} = \frac{p_1(x_n|\mathbf{x}_{n-1})}{p_0(x_n|\mathbf{x}_{n-1})} \prod_{i=1}^{n-1} \frac{p_1(x_i|\mathbf{x}_{i-1})}{p_0(x_i|\mathbf{x}_{i-1})}. \quad (12)$$

The log-likelihood ratio S_n becomes

$$S_n = \sum_{i=1}^{n} z_i = S_{n-1} + z_n \qquad z_n = \log\left(\frac{p_1(x_n|\mathbf{x}_{n-1})}{p_0(x_n|\mathbf{x}_{n-1})}\right), \quad (13)$$

where z_n is an increment of the log-likelihood ratio for each sample. The conditional probability density function under $\mathcal{H}_0$ can be written as

$$p_0(x_n|\mathbf{x}_{n-1}) = p_0(x_n) = \frac{1}{\pi\sigma^2} \exp\left(-\frac{|x_n|^2}{\sigma^2}\right), \quad (14)$$

and under $\mathcal{H}_1$

$$p_1(x_n|\mathbf{x}_{n-1}) = \frac{p_1(\mathbf{x}_n)}{p_1(\mathbf{x}_{n-1})} = \frac{1}{\pi} \frac{\det(\mathbf{R}_{n-1})}{\det(\mathbf{R}_n)} \times \exp\left(-\mathbf{x}_n^H \left(\mathbf{R}_n^{-1} - \begin{bmatrix} \mathbf{R}_{n-1}^{-1} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}\right) \mathbf{x}_n\right). \quad (15)$$

Since $\mathbf{R}_n$ is Hermitian, it can be rewritten as

$$\mathbf{R}_n = \begin{bmatrix} \mathbf{R}_{n-1} & \mathbf{r}_n \\ \mathbf{r}_n^H & r_0 \end{bmatrix}, \quad (16)$$

where $\mathbf{r}_n = [r_{-n+1}\ r_{-n+2}\ \cdots\ r_1]^T$ and $[\cdot]^T$ is matrix transpose. Using partitioned matrix inversion lemma [32], we obtain recursive relation between $\mathbf{R}_n^{-1}$ and $\mathbf{R}_{n-1}^{-1}$, i.e.,

$$\mathbf{R}_n^{-1} = \begin{bmatrix} \mathbf{R}_{n-1}^{-1} + \frac{1}{k_n} \mathbf{R}_{n-1}^{-1}\mathbf{r}_n\mathbf{r}_n^H \mathbf{R}_{n-1}^{-1} & -\frac{1}{k_n}\mathbf{R}_{n-1}^{-1}\mathbf{r}_n \\ -\frac{1}{k_n}\mathbf{r}_n^H \mathbf{R}_{n-1}^{-1} & \frac{1}{k_n} \end{bmatrix}, \quad (17)$$

where

$$k_n = \frac{\det(\mathbf{R}_n)}{\det(\mathbf{R}_{n-1})} = r_0 - \mathbf{r}_n^H \mathbf{R}_{n-1}^{-1}\mathbf{r}_n. \quad (18)$$

Using (18), the calculation of matrix determinant can be delivered recursively,

$$\det(\mathbf{R}_n) = k_n \times \det(\mathbf{R}_{n-1}). \quad (19)$$

Substituting (17) and (18) to (15), then we have

$$p_1(x_n|\mathbf{x}_{n-1}) = \frac{1}{\pi k_n} \exp\left(-\frac{1}{k_n} [\mathbf{x}_{n-1}^H\ x_n]\ \mathbf{R}_{n|n-1}^{-1} \begin{bmatrix} \mathbf{x}_{n-1} \\ x_n \end{bmatrix}\right), \quad (20)$$

with

$$\mathbf{R}_{n|n-1}^{-1} = \begin{bmatrix} \mathbf{R}_{n-1}^{-1}\mathbf{r}_n\mathbf{r}_n^H \mathbf{R}_{n-1}^{-1} & -\mathbf{R}_{n-1}^{-1}\mathbf{r}_n \\ -\mathbf{r}_n^H \mathbf{R}_{n-1}^{-1} & 1 \end{bmatrix}. \quad (21)$$

Furthermore, (20) can be written in a simpler form, i.e.,

$$p_1(x_n|\mathbf{x}_{n-1}) = \frac{1}{\pi k_n} \exp\left(-\frac{(x_n - \mu_n)^H (x_n - \mu_n)}{k_n}\right), \quad (22)$$

where

$$\mu_n = \mathbf{r}_n^H \mathbf{R}_{n-1}^{-1}\mathbf{x}_{n-1}, \quad (23)$$

and k_n can be obtained from (18). Equation (22) signifies that the conditional distribution of x_n under $\mathcal{H}_1$ is complex Gaussian with mean μ_n and variance k_n. In this case, x_n is a new sample and μ_n is deterministic since $\mathbf{x}_{n-1}$ is previously known. Each update in this scheme can be considered as to decide whether $(\boldsymbol{x}_n|\mathbf{x}_{n-1}) \sim \mathcal{CN}(\mu_n, k_n)$ or $(\boldsymbol{x}_n|\mathbf{x}_{n-1}) \sim \mathcal{CN}(0, \sigma^2)$. Thus, substituting (22) and (14) into (13), we can obtain the increment of the log-likelihood ratio,

$$z_n = \log\left(\frac{\sigma^2}{k_n}\right) - \left(\frac{|x_n - \mu_n|^2}{k_n} - \frac{|x_n|^2}{\sigma^2}\right). \quad (24)$$

The log-likelihood ratio in (13) is then updated with $S_n = S_{n-1} + z_n$.

TABEL I. ALGORITM FOR THE RECURSIVE SEQUENTIAL DETECTOR

Step	Action
Initialization	$n = 0$, choose α and β, determine the first row of $\mathbf{R}_{rn}$ and σ^2, $S_0 = 0$, and $\mathbf{R}_0^{-1}$ could be an empty matrix.
1)	Determine A dan B from (9)
Repeat:	
2)	Take a sample: $n = n + 1$
3)	Calculate the mean μ_n by using (23), the variance k_n by (18), det $(\mathbf{R}_n)$ by (19) and $\mathbf{R}_n^{-1}$ by (17)
4)	Calculate the increment of the log-likelihood ratio z_n using (24)
5)	Update the test statistic $S_n = S_{n-1} + z_n$
Until $S_n \geq \log(A)$ or $S_n \leq \log(B)$	
6)	If $S_n \geq \log(A)$, accept $\mathcal{H}_1$ (signal present), or If $S_n \leq \log(B)$, accept $\mathcal{H}_0$ (signal absent)

The algorithm for the recursive sequential detector can be seen in Table I. Note that the recursive method does not use approximation as in non-recursive sequential detector [29], so the performance in terms of probability of error and average sample number is expected to be the same as the conventional sequential detector. This is confirmed by simulations in section III.

C. Simulation Method

Simulation is used to measure the performance of the proposed recursive sequential detector. We compare the proposed method to the conventional sequential detector which directly calculates the inverse and determinant of the covariance matrix in (11) using Lower-Upper (LU) Decomposition and Gauss-Jordan elimination, respectively. The second reference that we use to compare to is the non-recursive sequential detector [29].

As the metrics, we use the actual probabilities of false alarm P_f and miss detection P_m, with the nominal probabilities of false alarm α and the nominal probability of miss detection β. Both are set to be equal (α = β) with values of $\alpha, \beta = 10^{-2}, 5 \times 10^{-2}$, and 10^{-1}. Another metric is the average sample number (ASN), under $\mathcal{H}_1$ when signal present (ASN_1)

and under $\mathcal{H}_0$ when signal absent (ASN_0). The overall performance is calculated based on 10^4 Monte-Carlo runs. The average received signal power $\overline{\mathrm{P}}_{\mathrm{r}}$ is assumed to be constant, so the SNR (in dB) is defined as

$$SNR = 10\log_{10}\frac{\bar{P}_r}{\sigma^2} = 10\log_{10}\frac{[\mathbf{R}_{rn}]_{ii}}{\sigma^2}, \quad \forall i = 1,2,\ldots \tag{25}$$

with the noise power $\sigma^2 = 1$. The signal covariance matrix can be rewritten as

$$\mathbf{R}_n = \gamma\Sigma_r + \mathbf{I}_n, \tag{26}$$

with $\gamma = 10^{\frac{SNR}{10}}$ and Σ_r is normalized signal covariance matrix. For all simulations, Σ_r is assumed to be

$$\Sigma_r = \begin{pmatrix} 1 & \rho_{-1} & \cdots & \rho_{-n+2} & \rho_{-n+1} \\ \rho_1 & 1 & \rho_{-1} & \cdots & \rho_{-n+2} \\ \vdots & \rho_1 & 1 & \ddots & \vdots \\ \rho_{n-2} & \vdots & \ddots & \ddots & \rho_{-1} \\ \rho_{n-1} & \rho_{n-2} & \cdots & \rho_1 & 1 \end{pmatrix}, \tag{27}$$

where $\rho_1 = \rho_{-1} = 0{,}6$, $\rho_2 = \rho_{-2} = 0{,}4$, $\rho_3 = \rho_{-3} = 0{,}2$, and $\rho_i = 0$ for $|i| > 3$.

Performance evaluation using simulation has the following objectives:

- Demonstrating the superiority of the recursive sequential detector over the quadrature detector in terms of the required average sample number to achieve the same target detection performance.
- Evaluating how much the performance of the recursive sequential detector differ from the non-recursive sequential detector.
- Verifying that the proposed recursive sequential detector has the same performances on P_f, P_m and ASN as the conventional sequential detector but it enjoys having much smaller computational complexities.

III. Results and Discussion

This section covers the simulation results using the setup described in the previous section. The ASN is utilized as a performance indicator to show that the proposed recursive sequential detector uses fewer samples than the quadrature detector, resulting in a faster detection process. The simulation results on the probabilities of false alarm and miss detection are then reviewed to demonstrate that the proposed detector can minimize computational complexity while maintaining these two performance criteria. This section concludes with a quantification of the computational complexity achievable by the proposed detector when compared to the existing detectors.

A. Average Sample Number and Probability of Error

Figure 1 shows the advantages of using a recursive sequential detector over a quadrature detector regarding the average sample number to achieve the same target error probabilities of α and β. For the overall nominal probabilities of α and β (where $\alpha = \beta$) and overall SNR ranges, the ASNs of the recursive sequential detector are significantly smaller (about half) than that of the quadrature detector. The gap between the two becomes more pronounced for lower SNRs since the sample number required by the quadrature detector rises exponentially faster than the recursive sequential detector. In some realizations, the sequential detector may require more samples than the quadrature detector, but the sequential detector requires much smaller on average. Note that the average sample number in Figure 1 is under $\mathcal{H}_1$ (ASN_1). The use of a recursive sequential detector is thus recommended over the quadrature detector, as it will increase the agility of the system in detecting the desired signal.

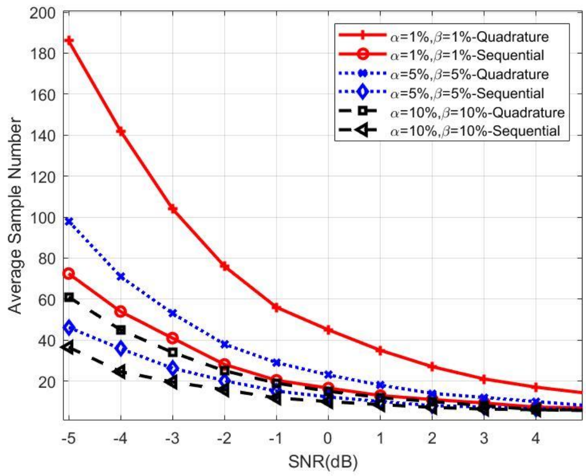

Figure 1. Required sample numbers for the sequential and quadrature detectors to achieve nominal α dan β

Table II shows the performances of the recursive sequential detector, the conventional sequential detector, and the non-recursive sequential detector. Based on the table, the three detectors have not significantly different performances. Some key points that can be highlighted from the table are:

1. The recursive sequential detector has the same actual decision errors of P_f and P_m as the conventional sequential detector. The non-recursive sequential detector is slightly larger P_f and slightly smaller P_m than the recursive sequential detector.
2. The recursive sequential detector has the same average sample numbers, ASN_1 and ASN_0 , as the conventional sequential detector. Whereas the non-recursive sequential detector has a slightly smaller ASN_1 (maximum 4.4 average samples at a low SNR -5 dB) and a slightly larger ASN_0 (maximum 2.8 average samples) than the recursive sequential detector.
3. For each nominal probabilities of α and β (where $\alpha = \beta$) and over all SNR ranges, the three sequential detectors generally satisfy inequality (10). Except for the non-recursive sequential detector at SNR 0 and 5 dB, P_f is larger than the target value. As an example, for $\alpha = \beta = 0{,}01$ and SNR = 5 dB, the three sequential detectors should satisfy the inequality $P_f \le \alpha/(1-\beta) = 0.01/(1-0.01) = 0.0101$. However, the non-recursive sequential detector does not satisfy the inequality, i.e., $P_f = 0.0659 > 0.0101$.

The difference in performance between recursive and non-recursive sequential detectors occurs due to the different approaches for reducing the complexity of calculating the invers and determinant of the signal covariance matrix $\mathbf{R}_n$. In the non-recursive method, $\mathbf{R}_n$ is first approximated by a circular matrix $\mathbf{C}_n$, which then leads to reduction of computational complexities for calculating the invers and determinant [24]. In contrast, the recursive method does not use an approximation, but we derive such that the calculation of the invers and determinant of the covariance matrix can be recursively implemented.

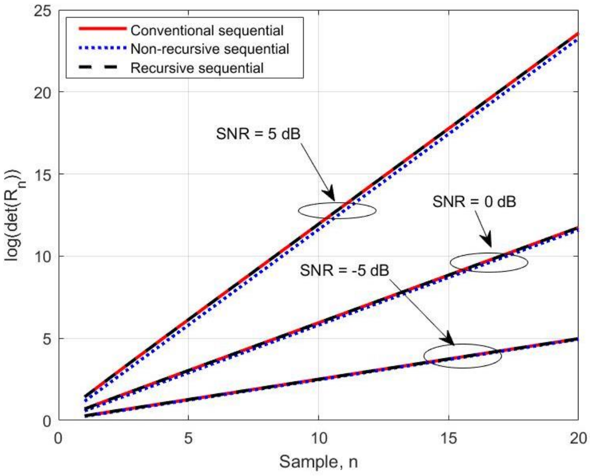

Figure 2. $\log(\det(\mathbf{R}_n))$ as the sample number n increasing, (conventional and recursive sequential entirely overlap)

Figures 2, 3, and 4 illustrate the consequences of having different approaches in the recursive and non-recursive sequential detectors. Note that, in these figures, the graphs for the conventional sequential detector and the recursive sequential detector overlap entirely, which might reduce their visibility.

Figure 2 shows the calculation of $\log(\det(\mathbf{R}_n))$ as the sample number n increases, for the recursive, conventional, and non-recursive sequential detectors. The values for the recursive and the conventional sequential detectors completely overlap as we expected, and the non-recursive sequential detector has slightly different values from the two others and becomes unnoticeable for very small SNR $= -5$ dB.

The way we calculate the inverse and determinants of the covariance matrix $\mathbf{R}_n$ will also impact the calculation of the log-likelihood ratio S_n. Figure 3 illustrates an instance of trajectories of the log-likelihood ratio S_n as a function of sample number n under $\mathcal{H}_1$ with SNR = -5 dB. The recursive and conventional sequential detectors have overlapped trajectories and the same stopping times $N_s = 58$. However, the trajectory for the non-recursive sequential detector is different, with the stopping time $N_s = 56$, producing different average sample numbers. Meanwhile, the three sequential detectors make correct decisions to favour $\mathcal{H}_1$ since they have the log-likelihood ratios $S_n \geq \log(A)$ at the stopping times. This condition occurs in most of the S_n trajectories. Thus, all three detectors satisfy the inequality of the actual probability of miss detection P_m in (10) for each $\alpha = \beta$, as shown in Table II.

Figure 4 illustrates the trajectories of the log-likelihood ratio S_n as the sample number n increases under $\mathcal{H}_0$. Figure 4.a shows for SNR = -5 dB and Figure 4.b for SNR = 5 dB. The figures depict the recursive sequential detector has trajectories completely overlap with the conventional sequential detector, with equal stopping times at $n = 41$ for SNR = -5 dB and at $n = 11$ for SNR = 5 dB. In addition, the recursive and conventional sequential detectors produce correct decisions under $\mathcal{H}_0$ for both SNRs since they have the log-likelihood ratios $S_n \leq \log(B)$ at the stopping times. This confirms the results in Table II which read that the performance of the recursive sequential detector is equal to the conventional sequential detector for the entire scenario.

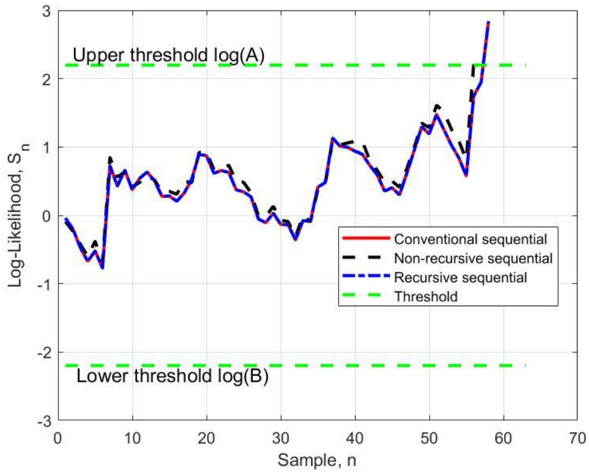

Figure 3. Trajectories for the log-likelihood ratio S_n as the sample number n increasing under $\mathcal{H}_1$ (SNR = -5 dB, $\alpha = \beta = 0.1$), (conventional and recursive sequential entirely overlap)

TABEL II. PERFOMANCES OF RECURSIVE SEQUENTIAL DETECTOR (D-R), CONVENTIONAL SEQUENTIAL DETECTOR (D-D), AND NON-RECURSIVE SEQUENTIAL DETECTOR (D-N)

Performance Metrics		SNR = -5 dB			SNR = 0 dB			SNR = 5 dB		
		$\alpha,\beta = 10^{-2}$	$\alpha,\beta = 5x10^{-2}$	$\alpha,\beta = 10^{-1}$	$\alpha,\beta = 10^{-2}$	$\alpha,\beta = 5x10^{-2}$	$\alpha,\beta = 10^{-1}$	$\alpha,\beta = 10^{-2}$	$\alpha,\beta = 5x10^{-2}$	$\alpha,\beta = 10^{-1}$
P_f	D-R	5.5 x 10^{-3}	3.2 x 10^{-2}	6.8 x 10^{-2}	4.7 x 10^{-3}	2.4 x 10^{-2}	5.1 x 10^{-2}	2.1 x 10^{-3}	1.2 x 10^{-2}	2.4 x 10^{-2}
	D-D	5.5 x 10^{-3}	3.2 x 10^{-2}	6.8 x 10^{-2}	4.7 x 10^{-3}	2.4 x 10^{-2}	5.1 x 10^{-2}	2.1 x 10^{-3}	1.2 x 10^{-2}	2.4 x 10^{-2}
	D-N	8.0 x 10^{-3}	4.2 x 10^{-2}	8.4 x 10^{-2}	1.7 x 10^{-2}	6.2 x 10^{-2}	1.0 x 10^{-1}	6.6 x 10^{-2}	1.3 x 10^{-1}	1.6 x 10^{-1}
P_m	D-R	8.5 x 10^{-3}	4.6 x 10^{-2}	9.3 x 10^{-2}	8.6 x 10^{-3}	4.5 x 10^{-2}	8.6 x 10^{-2}	7.8 x 10^{-3}	3.4 x 10^{-2}	7.6 x 10^{-2}
	D-D	8.5 x 10^{-3}	4.6 x 10^{-2}	9.3 x 10^{-2}	8.6 x 10^{-3}	4.5 x 10^{-2}	8.6 x 10^{-2}	7.8 x 10^{-3}	3.4 x 10^{-2}	7.6 x 10^{-2}
	D-N	7.5 x 10^{-3}	4.1 x 10^{-2}	8.4 x 10^{-2}	7.3 x 10^{-3}	3.6 x 10^{-2}	6.9 x 10^{-2}	3.3 x 10^{-3}	1.8 x 10^{-2}	3.6 x 10^{-2}
$\mathbf{ASN_1}$	D-R	73.8	46.7	33.5	15.4	10.5	8.2	4.8	3.6	3.0
	D-D	73.8	46.7	33.5	15.4	10.5	8.2	4.8	3.6	3.0
	D-N	69.4	44.0	31.7	13.4	9.3	7.4	4.1	3.4	3.1
$\mathbf{ASN_0}$	D-R	98.7	60.4	41.8	25.0	15.6	11.3	8.9	5.8	4.3
	D-D	98.7	60.4	41.8	25.0	15.6	11.3	8.9	5.8	4.3
	D-N	101.5	61.5	42.3	26.2	16.0	11.5	9.8	6.3	4.9

TABEL III. BIG-O NOTATION ($\mathcal{O}$) TO REPRESENT COMPUTATIONAL COMPLEXITIES FOR SEQUENTIAL DETECTORS

Calculation	Conventional	Non-Recursive	Recursive (proposed method)
Matrix Inverse $\mathbf{R}_n^{-1}$	Gauss-Jordan: $\mathcal{O}(n^3)$	$\mathcal{O}\ (n\log(n))$	Eq. 17: $\mathcal{O}(n^3)$
Matrix Multiplication	$\mathcal{O}(n^3)$	$\mathcal{O}(n^3)$	Eq. (18) & (23): $\mathcal{O}(n^2)$
Matrix Determinant, $\det(\mathbf{R}_n)$	$\mathrm{LU-Decomp}$: $\mathcal{O}(n^3)$	$\mathcal{O}(1)$	Unnecessary

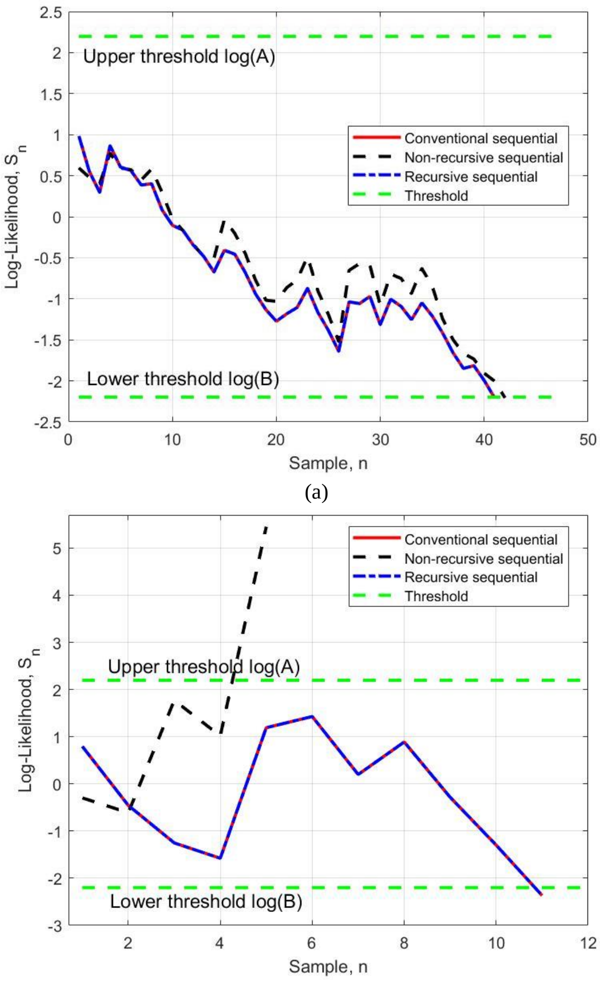

Figure 4. Trajectories for the log-likelihood ratio S_n as the sample number n increasing under $\mathcal{H}_0$: (a) SNR = -5 dB and (b) SNR = 5 dB, (conventional and recursive sequential entirely overlap)

The non-recursive sequential detector produces different trajectories S_n with the other two detectors. When the SNR is as small as -5 dB, the difference is not significant (<0.5 scale) as shown in Figure 4.a. However, when the SNR is as large as 5 dB, Figure 4.b shows that the non-recursive sequential detector has a very large trajectory gap which leads to false alarms. The phenomenon is more frequent for the non-recursive sequential detector, and it explains the reason that the actual probability of false alarm P_f does not satisfy inequality (10) for large SNR, such as 0 and 5 dB. To summarize, the Figures 2 and 3 provide the arguments for the key points 1 and 2 above, and Figure 4 for the key point 3.

B. Computational Complexity

To know the reduction in computational complexity offered by the recursive sequential detector, we can compare it with the conventional sequential detector that directly computes the inversion of the covariance matrix $\mathbf{R}_n^{-1}$ using Gauss-Jordan elimination and the determinant of the covariance matrix $\det(\mathbf{R}_n)$ using Lower-Upper (LU) Decomposition. Table III shows the order of complexity required for one iteration for the three sequential detectors.

The non-recursive sequential detector can significantly reduce the computational complexity of matrix inversion because it only needs to calculate one row of the invers matrix using FFT [29]. This cannot be obtained in the recursive sequential detector. A significant reduction is achieved by the recursive sequential detector when it calculates matrix multiplication, in which the non-recursive and the conventional sequential detectors have failed to achieve. Another advantage of using the recursive method is that there is no need to compute the determinant of the covariance matrix $\det(\mathbf{R}_n)$ since it has been replaced by faster calculation of k_n using (18).

To summarize, the recursive sequential detector has a lower computational complexity than the conventional one. However, the former maintains the same error probability and average sample number as the later. The recursive sequential detector has comparable computational complexities to the non-recursive sequential detector, but the former has better probabilities of false alarm than the latter. In terms of the probability of miss-detection, they both meet the target requirements. The three sequential detectors have no significant difference in the average sample numbers required to achieve the target probabilities of error under $\mathcal{H}_1$ as well as under $\mathcal{H}_0$.

These facts proof that the recursive sequential detector is a reliable alternative to use under correlated observations, particularly for the applications requiring high detection agility, such as in radar and cognitive radio. Nevertheless, the three detectors studied in this paper are based on simple hypotheses testing (known parameters). Our future work will derive the methods based on composite hypothesis testing, which are closer to the actual environment and might include recursive and non-recursive approaches.

IV. CONCLUSION

This paper presents a recursive method in sequential detector with the aim of reducing computational complexity under correlated observations. The recursive sequential detector derived based on the chain rule of probability has much lower computational complexities than the conventional sequential detector. At the same time, it still maintains equal average sample numbers and error probabilities. Moreover, the recursive sequential detector has comparable computational complexities to the non-recursive sequential detector with the advantage of having better probabilities of false alarm. Overall, the proposed recursive sequential detector offers more efficient solutions under correlated observations than the other two detectors. The three detectors investigated in this study rely on simple hypothesis testing. Our future work will construct methods based on composite hypothesis testing that are more relevant to the actual world, which may include both recursive and non-recursive approaches.

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