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Rigorous Mathematical Thinking: Conceptual Knowledge and Reasoning in the Case of Mathematical Proof

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Abstract

This study aims to analyze in-depth students' conceptual knowledge and reasoning when solving problems using mathematical proof as a rigorous mathematical thinking paradigm. The research uses a qualitative method with a case study approach that analyzes the mathematical proof ability of nine students who represent different cognitive functions from each level of rigorous mathematical thinking. The results showed that each level of rigorous mathematical thinking meant other indicators according to their ability to master conceptual knowledge and implement mathematical ideas through reasoning. This research has an impact on the treatment that the teacher must give in determining the learning model and evaluation instrument that can raise students' conceptual knowledge and reasoning.

Abstrak

Penelitian ini bertujuan untuk menganalisis secara mendalam pengetahuan konseptual dan penalaran siswa ketika memecahkan masalah menggunakan pembuktian matematis sebagai paradigma berpikir matematis yang ketat. Penelitian ini menggunakan metode kualitatif dengan pendekatan studi kasus yang menganalisis kemampuan pembuktian matematis sembilan siswa yang mewakili fungsi kognitif yang berbeda dari setiap tingkat pemikiran matematis yang teliti. Hasil penelitian menunjukkan bahwa setiap tingkat berpikir matematis yang teliti berarti indikator lain sesuai dengan kemampuannya untuk menguasai pengetahuan konseptual dan mengimplementasikan ide-ide matematika melalui penalaran. Penelitian ini berdampak pada perlakuan yang harus diberikan guru dalam menentukan model pembelajaran dan instrumen evaluasi yang dapat meningkatkan pengetahuan konseptual dan penalaran siswa.

Keywords: *Rigorous Mathematical Thinking; Conceptual Knowledge; Reasoning; Mathematical Proving.*

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INTRODUCTION

Students' cognitive function, basically, does not develop naturally, but needs to be actively developed through educational process (Lövdén et al., 2020). Teachers' hard work in determining models and various learning instruments determines formation of students' cognitive function (Esterhuizen, 2014), considering a notion that suitability of student learning conditions with learning designed by teacher is able to construct conceptual thinking and reasoning patterns. Construction of conceptual mindset and reasoning is closely related to learning Mathematics materials since Mathematics fundamentally focuses on quality of thought and understanding of reason in obtaining a set of knowledge (Rocess & Ecurrent, 2018). As an attempt to obtain quality knowledge on learning mathematics, ability to have rigorous mathematical thinking in understanding important mathematical ideas is very necessary. It is in the form of mathematical concepts and principles, as well as an in-depth understanding of relationship between these concepts and principles. Quality of thinking at stages of rigorous mathematical thinking positions students to compile a high level of accuracy about clarity and completeness of concepts and definitions, involvement of critical and logical thinking, and involvement of deep understanding between critical thinking and logic patterns (Kinard, 2006; Kinard & Kozulin, 2015).

Importantly, rigorous mathematical thinking through two underlying psychological theoretical approaches, namely Vygotsky's theory and Feuerstein's concept, allows for active construction of conceptual knowledge and reasoning (Kinard, 2006; Kinard & Kozulin, 2008). This is because those two experts emphasize learning to improve students' cognitive function (Kozulin ,2002). Principally, it is in accordance with a principle of learning mathematics that students must be able to build conceptual knowledge and interpret reasoning in solving problems so that their cognitive function is well developed. Conceptual knowledge refers to knowledge of concepts, including principles and definitions (Jon, 2013). Conceptual knowledge is the knowledge that can link several pieces of knowledge (Hurrell, 2021). For this reason, conceptual knowledge is a foundation in describing students' ability to understand mathematics materials, while reasoning refers to relationship and integration in students' interactive learning activities regarding knowledge of basics of mathematics and mathematical components (Wilkinson et al., 2018). Reasoning also describes decision-making process of students' thinking skills (Erdem & Gürbüz, 2014; Gürbüz & Erdem, 2016)

In mathematics learning, understanding concept builds students' knowledge, one of which is through proof so that students can learn well (Haji & Yumiati, 2019). Mathematical proof is an essential part of mathematics, especially regarding the construction of the proof (Alpi & Evans, 2019; Syamsuri et al., 2018). Most of the mathematical proof steps apply the basic rules of logic (Krantz, 2007). Besides, a good method to practice writing proof can be done through reasoning since it is involved as long as students carry out mathematical proof process (Stefanowicz et al., 2014). In addition, reasoning affects a person's ability to think logically, critically, and systematically (Maidiyah et al., 2021). Thus, understanding concept that is included in conceptual knowledge and students' thinking process that is included in reasoning are two important elements in mathematical proof. However, proof is a severe matter in determining school curriculum in various countries (Noto et al., 2019). Accordingly, it is necessary to dig deep related students' conceptual knowledge and reasoning in cases of mathematical proof. This is because in mathematics learning, case of mathematical proof occupies the highest level in representing abstraction as the essence of mathematics (Lingefjärd & Hatami, 2020).

Several studies have identified importance of in-depth analysis for students' conceptual knowledge and reasoning in mathematics learning (Al-Mutawah et al., 2019; Gilmore et al., 2018; Nurjanah et al., 2020). Previous research stated that most of the students were weak in conceptual knowledge. These studies emphasize analyzing students' conceptual understanding of mathematical problems for basic geometry and arithmetic materials. While some sources show empirical evidence which states that weak mathematical reasoning is one of the leading causes of student learning difficulties (Arshad et al., 2017; Dollo, 2018). In fact, there is no study that analyzes elements of conceptual knowledge and reasoning in depth and simultaneously on mathematical proof. Several studies related to mathematical proof cases are more directed at limitations of students in solving mathematical proof and difficulties in finding initial ideas (CadwalladerOlsker, 2011; Gilmore et al., 2018; Hanna & Knipping, 2020; Rocha, 2019), while studies on rigorous mathematical thinking are more related to conceptual understanding (Kinard & Kozulin, 2008; Nugraheni et al., 2018).

Furthermore, mathematical thinking paradigm is strictly in accordance with current curriculum objectives in Indonesia, in which learning, especially Mathematics, must be based on Higher Order Thinking Skills (HOTS). Consequently, students' skills should lead to thinking at the highest level (Fuady, 2016). Various supporting components of learning that accommodate students' higher order thinking skills are options whose application is an obligation for classroom teachers (Darling-Hammond et al., 2020). Accordingly, it is necessary to conduct analysis on these supporting components, including conceptual knowledge and reasoning from the context of rigorous mathematical thinking to determine students' thinking skills so that teachers can provide appropriate treatment to students and choose various learning strategies that are appropriate to students' conditions and cognitive skills. This kind of thinking is essential considering the further students move from maximizing their thinking skills to being more focused on getting results. Learning needs to include components of conceptual knowledge and reasoning because conceptual knowledge is an important element of reasoning that is useful to show an extent to which students can reason well (Lyons, 2014).

Based on the explanations, this study analyzed context of rigorous mathematical thinking in terms of students' conceptual knowledge and reasoning in the case of mathematical proof. Results of this study can be a basis for teachers' rationale to create a concept of didactic situations which are able to develop students' higher-order thinking skills so that mathematics learning returns to its scientific context and not only memorizing and understanding but also developing skills in linking between materials, compiling precise mathematical proof, and explaining various reasons related to prevailing concept. Teachers must also facilitate ways to develop conceptual knowledge and reasoning of students with their different levels of thinking.

METHOD

This research uses a qualitative method with a case study approach. The case study approach is a research focus that can identify critical factors, processes, and relationships involved in a phenomenon (Harrison et al., 2017; Rashid et al., 2019). The phenomenon in this study is that the ability to think mathematically is strictly an essential phenomenon in learning which is reviewed in the context of conceptual knowledge and reasoning in the case of mathematical proof. The form of representation of strict mathematical thinking in the case of mathematical proof is in the form of a rigorous mathematical thinking test instrument and interview guidelines. The two instruments function to obtain data on the completeness of research subjects in representing their strict mathematical thinking.

The research subjects were three students from the Mathematics Education Study Program at a well-known private university in Cirebon. The research subjects went through a random selection process from the results of the initial test of Real Numbers material representing each level of cognitive function from rigorous mathematical thinking. The three research subjects represent three mental function levels groups: qualitative, quantitative with precision, and relational abstract. Then the research subjects carried out tests for problems involving mathematical proof and then analysed their conceptual knowledge and reasoning. The analysis of the test results adjusts to the indicators of each level of rigorous mathematical thinking, conceptual understanding, and reasoning, which also adapts to the indicators of mathematical proof. After obtaining the results of the Real Numbers material test, the research subjects received several questions to represent their learning outcomes and learning experiences regarding the mathematical proof. Based on previous interviews, the researchers obtained firmness regarding the conceptual abilities and mathematical reasoning of each research subject.

RESULT AND DICUSSION

Result

In this study, data displayed are qualitative data. Students took Real Numbers material test for mathematical proof cases. After checking their answers, three groups were formed, each group representing three levels of cognitive function of rigorous mathematical thinking. From each group, one student's answer was analyzed based on indicators of conceptual knowledge and reasoning that adjusted to indicators of each level of rigorous mathematical thinking. Figure 1 shows answers of a student from qualitative thinking group.

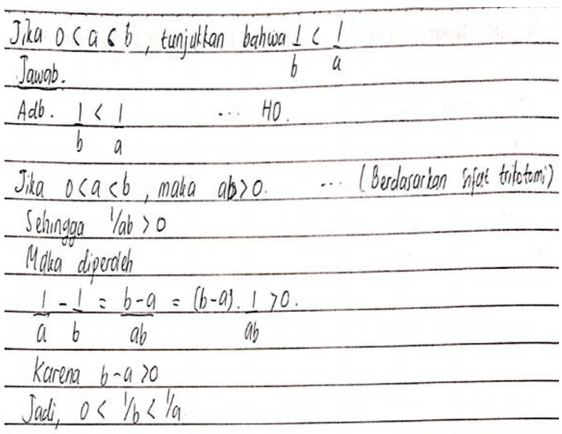

Figure 1. A Student with Qualitative Thinking Level

Based on results of work in Figure 1, the student presented a form of proof by constructing more than one source of information, by applying several principles and concepts. He/she was able to collect initial ideas and sort answers clearly and completely and was able to compile code answers systematically. Thus, it can be concluded that this student was in a category of cognitive function of qualitative thinking.

Related to conceptual knowledge,

someone with qualitative thinking type can connect general principles of knowledge regarding mathematical proof by adhering to process of proof, giving reasons for given process of proof, and providing an overview of number symbols included in process of proof. The second indicator shows that student had knowledge of general principles that clearly underlay mathematical proof procedure. He/she was able to connect every concept related and suitable to be applied in proving problem, but it was still limited to their understanding of general concepts and not yet led to development of his/her knowledge. Referring to reasoning indicators for students with qualitative thinking, he/she was able to draw logical conclusions. Thus, based on this student understanding, written proof process already reached proven criteria. He/she was also able to estimate answer by following pattern of relationships from various mathematical situations that he/she had previously predicted based on initial hypothesis, such as relationship between numbers a, b, ab, nature of real numbers, and operations used, so valid proof could be arranged.

Moreover, the second student was in level of quantitative thinking with precision. Figure 2 presents his/her test result.

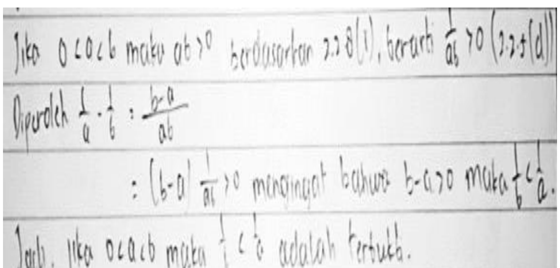

Figure 2. A Student with Quantitative Thinking with Precision Level

Based on result presented in Figure 2, the student was already able to identify and describe through pattern of proof of relationship between concepts that had been studied previously, regarding concept of difference in fractions and inequalities. He/she was also able to analyze, organize, and represent each process of proof based on relationship between theorem and prevailing operations as well as relationship between concept of number operations. Besides, he/she was able to solve a case of inequality that applied to a and b, then put it back together for proof through a number operation procedure on fractional inequality.

Additionally, in terms of conceptual knowledge, student representative in this group knew general principles of proof, so that process of proof in Figure 2 displays definition of positive real numbers, even from the time he/she presented his/her initial ideas. Although appearance of process of proof was not neat since there were no complete reasons, principles underlying mathematical proof had been successfully demonstrated. For reasoning component, logical conclusions and brief explanations related to process of proof could be displayed, completed with estimated answers as well as closing sentence of proof. The student was able to compile direct proof according to his/her understanding of concepts related to theorems in sequence and properties of real number, so that a valid proof was constructed. The third student with cognitive function at level of abstract relational thinking had a systematic answer as shown in Figure 3.

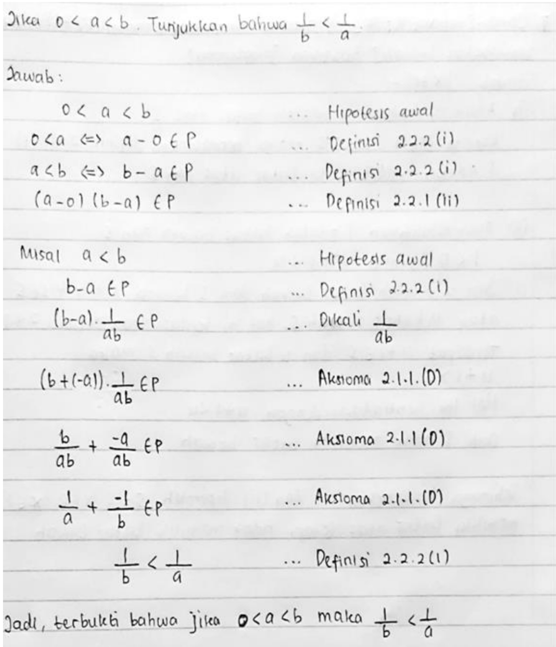

Figure 3. A Student with Abstract Relational Thinking Level

Figure 3 demonstrates that the student was able to articulate evidence in a logical mathematical manner; therefore, it seems clear that his/her process of thinking and proof was consistent and systematic. Information and reasons in each step of proof were neat, valid, and logical. Subject was able to connect each process of proof with definitions, theorems, axioms, and operating procedures that applied to sequence and properties of real number. In this case, construction of previous concepts related to inequality in fractions and natural numbers with properties and sequences of real numbers was well connected, thus forming a structured conclusion and mathematically inductive to deductive mindset. This student analyzed and reflected in detail to describe all cognitive activities clearly.

More importantly, subject at abstract relational thinking level was able to identify facts from known components in the problem and described them according to his/her understanding of previous concepts for proof. Symbols of inequality could be deciphered precisely, while initial ideas were developed perfectly according to concepts and principles of properties and sequences of real numbers. The subject was conceptually able to represent answer so that the problem was solved directly according to pattern of proof. Besides, about reasoning, students are able to draw logical conclusions from evidence that had been systematically arranged. Based on initial ideas and solutions found, it seems clear that process of proof directly linked new experience and knowledge. This student formulated valid conclusions by providing closing sentence answering objective of question given.

Discussion

The results showed that the conceptual knowledge and reasoning of the three research subjects represented their ability to understand the Real Numbers material. These results obtained reinforcement from interviews on the three subjects. Each student can channel each cognitive stage that reflects the sequence of answers from mathematical proofs. Students with qualitative thinking levels for rigour mathematical thinking show that they can implement solutions well but cannot represent them in structured mathematical proof answers. Students can understand the meaning of the questions, but they are still confused when compiling a series of solutions. Based on the interviews with students, he encountered difficulties when trying to relate some material by the components in the question, resulting in the answers being inconsistent with the correct systematics.

Meanwhile, students at the level of quantitative thinking with precision can relate the previous mathematical concepts to the need for answers. Still, the results they write have not shown a perfect mathematical proof structure. From the interviews, students were indeed able to write down the various components in the answers, only having difficulties writing down the interrelationships between the materials to apply them in a complete answer structure. Students with abstract relational thinking levels can construct a perfect mathematical proof structure with the right reasons for each proof step. The interview results with the subject showed that he was able to offer the completeness of the answer structure. He can also draw some material related to the components in the problem and its conclusions. The thoughts of the three students when working on mathematical proof problems showed the correct relationship between conceptual knowledge, reasoning, and rigour of mathematical thinking. According to Letuna, Natalia, and Resureicao (Letuna et al., 2020), conceptual understanding can develop in learning that prioritizes rigorous mathematical thinking. Meanwhile, according to Aulia and Fitriyani (2019), the reasoning is part of rigour mathematical thinking which is a continuation of cognitive processes at a concrete level

Several studies have shown that rigorous mathematical thinking is an essential element related to concepts and reasoning. Implementation of rigorous mathematical thinking in students involves construction of mathematical concepts using three cognitive stages, namely cognitive development, content development as a process, and cognitive conceptual construction practices (Dayat Hidayat et al., 2021). Learning practice in this case provides flexibility for students to construct new experiences and knowledge they acquire, while teacher becomes a mediator in utilizing cognitive functions of students in solving math problems. Building mathematical thinking skills of students can be done by getting them used to solving problems that are closely related to mathematical accuracy, for example through mathematical proof using idea of reasoning (Hamami, 2014). However, returning to the differences in student absorption in understanding knowledge, conceptual experience, and student reasoning must also be a concern for teachers. The teacher's attention to these two components is to respond and appreciate students' learning abilities.

Unfortunately, some studies have failed to align theoretical claims about nature of conceptual knowledge with tasks measuring knowledge. However, results of a study conducted by Crooks and Alibali (Crooks & Alibali, 2014) specifically suggest two types of conceptual knowledge, namely knowledge of general principles and knowledge of principles underlying procedures. These two types of conceptual knowledge become additional input in discussion of this study, so that results of analysis do not go out of context. In this study, answers of the three research subjects are in accordance with descriptions of conceptual knowledge indicators, in which their general flow of thought and peculiarities of each are clear. In addition, results of a study by Agustyaningrum, Hanggara, Husna, Abadi, and Mahmudi (Agustyaningrum et al., 2019) claimed that reasoning is important to build mathematical concepts of students. Reasoning is a foundation of mathematics (Kollosche, 2021); therefore, it is important to study and discuss building blocks of reasoning in elementary, middle, and tertiary education level students since learning mathematics and reasoning are inseparable (Jones & McLean, 2018).

In terms of concepts and reasoning in learning mathematics, mathematical proof is one case as the major focus. Results of this study showed that when research subject was at level of rigorous mathematical thinking for abstract relational, indicators of conceptual knowledge and reasoning were fulfilled, but students sometimes missed writing conditions of numbers as part of process of proof. Here is one of the examples: a condition of a,b ∈Z,b≠0 was not written for a fraction a/b. Subject tried to understand meaning of problem, then used his/her long-term memory to remember various problems that had been previously discussed based on previous experience. He/she then linked ideas to definitions, theorems, axioms, and principles applying to properties and sequences of real numbers. This shows that the student had activated entire cognitive domain when learning new material. Here, objectives of learning mathematics in accordance with curriculum and desire for formation of higher order thinking skills as an important aspect of learning can be fulfilled (Tanujaya et al., 2017)

When subjects are at level of qualitative and quantitative thinking with precision, they are basically able to relate experience and new material obtained with their understanding of solution to a problem. Their strength to represent answers adjusted to various definitions, theorems, axioms, and principles in properties of real numbers material has reached a good stage presenting correct answers until final conclusions are obtained. However, accuracy, tidiness of presentation, systematic answers, and in-depth analysis of flow of thought as a focus of mathematical proof are still not fully presented. Accordingly, it is necessary to have a learning atmosphere that prioritizes discussion and gives practice questions related to mathematical proof as an attempt to increase and develop level of cognitive function of students' rigorous mathematical thinking skills in order to create conceptual, logical, and valid mathematics learning (Letuna et al., 2020). Because based on the results of the research, students can understand the meaning of the questions and arrange ideas and materials related to the answers, only the difference appears when they start to compose sentences of mathematical proof, there are indeed difficulties that arise in that section, and some are already able to arrange systematically.

Learning by prioritizing the honesty factor of students in presenting answers according to their level of cognitive thinking and the diversity of mindsets in understanding concepts and reasoning abilities must be one of the main benchmarks in the success of the mathematics learning process. Students are independent learners and are accustomed to linking learning experiences or concepts. Those concepts from materials received in previous learning already have conceptual benchmarks that they will use to solve mathematical problems they encounter. Only sometimes for some students, the difficulty factor in finding the initial idea of solving mathematical proof or errors in representing the relationship between materials that should be a solution makes the mathematical proof problem model a scourge for them. So that one that can be a solution to this problem is to first build students' conceptual abilities and mathematical reasoning through preliminary questions that provoke creativity in answering questions. Give students several light mathematical problems, then move to moderate levels until they end in more severe difficulties so that students are accustomed to solving cases with their mindset.

From this study, several innovations in mathematical proof related to conceptual knowledge and reasoning emerged along with students' analytical skills from higher-order thinking levels. These innovations include students' brain activity being able to harmonize experiences, new material, and challenges in the questions to then arrange mathematical proofs that represent the structure of a person's cognitive function. Learning conditions that do not channel students' thinking abilities have an unfavourable effect. Cognitive abilities become less honed to the maximum and give the right outcome. Cognitive function has an important position in solving mathematical proof cases; one way is through conceptual knowledge and reasoning activities. Another innovation is giving evaluation questions based on concepts and logic, allowing students to channel their creative ideas that adapt to their previous understanding. The innovations that appear in this study mean that students can maximize their mathematical thinking skills if the teacher can design instruments suitable for students' abilities.

CONCLUSSION

Each level of cognitive function of rigorous mathematical thinking provides its uniqueness in representing students' conceptual knowledge and reasoning in solving mathematical proof problems. Subjects at the level of qualitative thinking can compose direct proofs by displaying number symbols and applying general principles that underlie mathematical proofs according to the understanding of each student—drawing valid and straightforward conclusions. Subjects at the level of quantitative thinking with precision can arrange a direct line of evidence clearly and concisely by connecting between experience and new material so that the proof is valid. Students formulate initial ideas and then relate them to theorems that focus on the specifics of the real number system material to draw valid conclusions. For students with abstract relational thinking, the preparation of direct evidence looks complete with the answer flow. Beginning of the answer from the implementation of the basic concepts of fractions with clear reasons to make valid and logical conclusions. The focus conclusion sentence follows the purpose of the question without giving multiple interpretations.

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