Phys. Comm. 7 (1) 2023 : 22 - 27

Physics Communication

http://journal.unnes.ac.id/nju/index.php/pc

Lagrangian Equation of Coupled Spring-Pendulum System

Nur Widya Rini✉, Joko Saefan, Nur Khoiri
Department of Physics Education, Universitas PGRI Semarang, Semarang, Indonesia

Article Info

Article history:
Submitted 10 December 2022
Revised 01 February 2023
Accepted 03 February 2023

Keywords:
lagrangian, equation of motion, dynamics

Abstract

A coupled spring-pendulum system in a conservative field was studied where the equation of motion of the system using Lagrangian and Hamiltonian equation were obtained. The equation of motion represented by a second-order differential equation from the three generalized coordinate were used. The potential energy equal to zero when the system is in its equilibrium position. The generalized coordinate that being used were the angle of the first pendulum θ_1, the angle of the second pendulum θ_2, and the increase in the length of the spring x. The resulting equation of motion can be used to determine the dynamics behavior of the system at any time. Students' understanding is expected to be more complete by providing a procedure to derive the Lagrangian equation of motion.

Correspondence:
Department of Physics Education, Universitas PGRI Semarang
Jl. Lontar No. 1 Semarang, Kota Semarang, Indonesia 50125
e-mail: rininurwidya@gmail.com

p-ISSN 2528-5971
e-ISSN 2528-598X

INTRODUCTION

In this paper, the equation of motion of a complex mechanical system have been derived and solved approximately. The energy of the system is used to get the Lagrangian equation of motion of the system. Then, the system was solved using the Euler-Lagrange equation of motion. It is well known that physical systems can be described by their Lagrangian, and from the Lagrangian function one can obtain second order differential equations of motion describing such dynamic systems (Nenuwe, 2019).

From a historical perspective, Newton's Second Law is fundamental, much research having been derived from it with the introduction of more elaborate principles and demands. In addition, the Lagrange equations of motion have an exceptional and important property, namely that they preserve their form in any system of coordinates, even moving ones (de la Peña, 2020). The motion of the pendulum depends on its total energy $E = T + V$ (Rafat, 2009).

There are many previous papers that analyze the mechanical system and Lagrangian equation of motion. This paper explained the Lagrangian equation of motion for a 2D double spring-pendulum (Nenuwe, 2019). He analytically obtained the equation of the motion and numerically solved the equation using Maple. Another paper also analyzed and explained how the Lagrange and the Hamiltonian equations of motions are derived for the double pendulum system (Elbori & Abdalsmd, 2017). Furthermore, they compare the numerical solution and simulation, and change of angular velocities with time for certain system parameters at varying initial conditions (Elbori & Abdalsmd, 2017). Moreover, Biglari & Jami (2016), shows their attempts to numerically analyze the double pendulum system and examined the motion leads to chaotic feature. In addition, Rafat (2009) also investigate an idealized model of double square pendulum by first obtain the equations of motion using the Lagrangian formalism.

The system analyzed in this paper was a coupled spring-pendulum system as described in Figure 1. This dynamics system consists of a pendulum with mass m_1 connected to an inextensible rod l_1 that assumed its mass to be extremely small. Then, the pendulum is connected to a spring with a mass m_2 attached at the end of the spring. θ_1 is the swing angle between the rod and vertical line and θ_2 is the swing angle between the spring and vertical line. The unstretched length of the springs is l_2 and the spring extends the length by x. It is assumed that the spring has a very small mass that we can neglect. The interest of this study is to analytically derive the equation of motion for the coupled spring-pendulum system.

METHODS

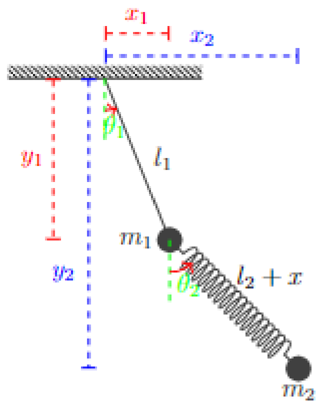

Figure 1. Coupled spring-pendulum system.

The coordinates θ_1, θ_2, and x were chosen as generalized coordinates. The equations of motion as second order differential equation can be obtained from those generalized chosen coordinates. The interest in this study is theoretically obtain the equations of motion for pendulum-spring system with time dependent variable. In this case, the Lagrangian of the system is described by the kinetic energy and the potential energy as given by

$$L = T - V \tag{1}$$

with T is the kinetic energy and V is the potential energy of the system (Rafat, 2009). Then, we insert the Lagrangian equation into Euler-Lagrange as follows

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{2}$$

with q is the generalized coordinates of the system. Furthermore, the Euler-Lagrange equations have been solved to find the equation of motion. The positions of m_1 and m_2 at any time in the system are expressed in Cartesian coordinates, namely (x_1, y_1) and (x_2, y_2).

RESULT AND DISCUSSION

The first step to derive the equation of motion is defining the kinetic and potential energy of the system. The kinetic energy of the system is obtained from

$$T = \frac{1}{2}m_1(\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2}m_2(\dot{x_2}^2 + \dot{y_2}^2) \tag{3}$$

with

$$x_1 = l_1 \sin\theta_1 \tag{4}$$
$$x_2 = \sin\theta_1 + (l_2 + x)\sin\theta_2 \tag{5}$$
$$y_1 = -l_1 \cos\theta_1 \tag{6}$$
$$y_2 = -l_1 \cos\theta_1 - (l_2 + x)\cos\theta_2. \tag{7}$$

Then the derivative of the variable x_1, x_2, y_1 and y_2 with respect to time is obtained as

$$\dot{x}_1 = l_1 \cos\theta_1 \dot{\theta}_1 \tag{8}$$

$$\dot{x}_2 = l_1 \cos\theta_1 \dot{\theta}_1 + (l_2 + x)\cos\theta_2 \dot{\theta}_2 + \dot{x}\sin\theta_2 \tag{9}$$

$$\dot{y}_1 = l_1 \sin\theta_1 \dot{\theta}_1 \tag{10}$$

$$\dot{y}_2 = l_1 \sin\theta_1 \dot{\theta}_1 + (l_2 + x)\sin\theta_2 \dot{\theta}_2 - \dot{x}\cos\theta_2 \tag{11}$$

so, we get

$$\dot{x}_1^{\,2} = l_1^2 \cos^2\theta_1 \dot{\theta}_1^{\,2} \tag{12}$$

$$\dot{x}_2^{\,2} = l_1^2 \cos^2\theta_1 \dot{\theta}_1^{\,2} + (l_2 + x)^2 \cos^2\theta_2 \dot{\theta}_2^{\,2} + \dot{x}^2 \sin^2\theta_2 + 2l_1(l_2 + x)\cos\theta_1 \cos\theta_2 \dot{\theta}_1\dot{\theta}_2 + 2l_1 \sin\theta_2 \cos\theta_1 \dot{\theta}_1\dot{x} + 2(l_2 + x)\sin\theta_2 \cos\theta_2 \dot{\theta}_2\dot{x} \tag{13}$$

$$\dot{y}_1^{\,2} = l_1^2 \sin^2\theta_1 \dot{\theta}_1^{\,2} \tag{14}$$

$$\dot{y}_2^{\,2} = l_1^2 \sin^2\theta_1 \dot{\theta}_1^{\,2} + (l_2 + x)^2 \sin^2\theta_2 \dot{\theta}_2^{\,2} + \cos^2\theta_2 \dot{x} + 2l_1(l_2 + x)\sin\theta_1 \sin\theta_2 \dot{\theta}_1\dot{\theta}_2 - 2l_1 \sin\theta_1 \cos\theta_2 \dot{\theta}_1 - 2(l_2 + x)\sin\theta_2 \cos\theta_2 \dot{\theta}_2\dot{x}. \tag{15}$$

The velocity of the first pendulum is obatined by adding up the equation (12) and (4)

$$\dot{x}_1^{\,2} + \dot{y}_1^{\,2} = l_1^2 \dot{\theta}_1^{\,2}. \tag{16}$$

Then, the velocity of the second pendulum is obtained by adding up the equation (13) and (15)

$$\dot{x}_2^{\,2} + \dot{y}_2^{\,2} = l_1^2 \dot{\theta}_1^{\,2} + (l_2 + x)^2 \dot{\theta}_2^{\,2} + \dot{x}^2 + 2l_1(l_2 + x)\cos(\theta_1 - \theta_2) - 2l_1 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{x}. \tag{17}$$

The equation for the total kinetic energy is obtained in equation (3) as followed

$$T = \frac{1}{2}m_1\left(l_1^2\dot{\theta}_1^{\,2}\right) + \frac{1}{2}m_2\left(l_1^2 \dot{\theta}_1^{\,2} + (l_2 + x)^2 \dot{\theta}_2^{\,2} + \dot{x}^2 + 2l_1(l_2 + x)\cos(\theta_1 - \theta_2) - 2l_1 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{x}\right). \tag{18}$$

The form of potential energy can be found with the assumption that the potential value is zero attached to m_1 so that the potential form of the coupled spring-pendulum system is the sum of the first mass m_1, the second mass potential m_2 and the spring potential terms. We get the total of potential as

$$V = -(m_1 + m_2)gl_1 \cos\theta_1 - m_2 g(l_2 + x)\cos\theta_2 + \tfrac{1}{2}kx^2. \tag{19}$$

The Lagrangian L of the system defined in equation (1). Therefore, L of the coupled spring-pendulum system defined according to

$$L = \frac{1}{2}m_1\left(l_1^2\dot{\theta}_1^{\,2}\right) + \frac{1}{2}m_2\left(l_1^2 \dot{\theta}_1^{\,2} + (l_2 + x)^2 \dot{\theta}_2^{\,2} + \dot{x}^2 + 2l_1(l_2 + x)\cos(\theta_1 - \theta_2) - 2l_1 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{x}\right) + (m_1 + m_2)gl_1 \cos\theta_1 + m_2 g(l_2 + x)\cos\theta_2 - \frac{1}{2}kx^2 \tag{20}$$

The Euler-Lagrange equation of motion in chosen generalized coordinates as written in equation (2) decomposed into the following equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \tag{21}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0 \tag{22}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{23}$$

Based on equation (21), from equation (20) we can obtain expression as

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 (l_2 + x)\sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 - m_2 l_1 \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{x} - (m_1 + m_2)gl_1 \sin\theta_1 \tag{24}$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2)l_1^2\dot{\theta}_1 + m_2 l_1 (l_2 + x)\cos(\theta_1 - \theta_2)\dot{\theta}_2 - m_2 l_1 \sin(\theta_1 - \theta_2)\dot{x}. \tag{25}$$

Since θ_1, θ_2, and x are functions of time, yield

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 \cos(\theta_1-\theta_2)\,\dot{\theta}_2\dot{x} + m_2 l_1 (l_2 + x)\cos(\theta_1-\theta_2)\,\ddot{\theta}_2 - m_2 l_1 (l_2 + x)\sin(\theta_1-\theta_2)\left(\dot{\theta}_1 - \dot{\theta}_2\right)\dot{\theta}_2 - m_2 l_1 \sin(\theta_1-\theta_2)\,\ddot{x} - m_2 l_1 \cos(\theta_1-\theta_2)\,(\dot{\theta}_1 - \dot{\theta}_2)\dot{x} \quad (26)$$

So that the Euler-Lagrange equation (21) becomes:

$$(m_1+m_2)l_1\ddot{\theta}_1 = -m_2(l_2 + x)\cos(\theta_1-\theta_2)\,\ddot{\theta}_2 + m_2\sin(\theta_1-\theta_2)\,\ddot{x} - m_2(l_2 + x)\sin(\theta_1-\theta_2)\,\dot{\theta}_2^{\,2} - 2m_2\cos(\theta_! - \theta_2)\,\dot{\theta}_2\dot{x} - (m_1+m_2)g\sin\theta_1. \quad (27)$$

For solving the Euler-Lagrange equation in general coordinates θ_2, equation (22) is obtained in the same steps, as follows

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 (l_2+x)\sin(\theta_1-\theta_2)\,\dot{\theta}_1\dot{\theta}_2 + m_2 l_1\cos(\theta_1-\theta_1)\,\dot{\theta}_1\dot{x} - m_2 g(l_2+x)\sin\theta_2 \quad (28)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2(l_2 \dot{+} x)^2\dot{\theta}_2 + m_2 l_1 (l_2+x)\cos(\theta_1 - \theta_2)\,\dot{\theta}_1. \quad (29)$$

Since θ_1, θ_2, and x are functions of time, yield

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2(l_2+x)^2\ddot{\theta}_2 + 2m_2(l_2+x)\dot{\theta}_2\dot{x} - m_2 l_1(l_2 + x)\sin(\theta_1-\theta_2)\left(\dot{\theta}_1 - \dot{\theta}_2\right)\dot{\theta}_1 + m_2 l_1(l_2 + x)\cos(\theta_1-\theta_2)\,\ddot{\theta}_1 + m_2 l_1\cos(\theta_1-\theta_2)\,\dot{\theta}_1\dot{x}. \quad (30)$$

So that the Euler-Lagrange equation (22) becomes:

$$(l_2+x)\ddot{\theta}_2 = -l_1\cos(\theta_1-\theta_2)\,\ddot{\theta}_1 - 2\dot{\theta}_2\dot{x} + l_1\sin(\theta_1-\theta_2)\,\dot{\theta}_1^{\,2} - g\sin\theta_2. \quad (31)$$

To fulfill the equation (27),s variable $\ddot{\theta}_2$ multiplied by $-m_2\cos(\theta_1-\theta_2)$, then we obtained the expression

$$-m_2(l_2+x)\cos(\theta_1-\theta_2)\,\ddot{\theta}_2 = m_2 l_1\cos^2(\theta_1-\theta_2)\,\ddot{\theta}_1 - m_2 l_1\sin(\theta_1-\theta_2)\cos(\theta_1 - \theta_2)\left(\dot{\theta}_1-\dot{\theta}_2\right)\dot{\theta}_1 - m_2 l_1\sin(\theta_1-\theta_2)\cos(\theta_1 - \theta_2)\,\dot{\theta}_1\dot{\theta}_2 + 2m_2\cos(\theta_1-\theta_2)\,\dot{\theta}_2\dot{x} + m_2 g\sin\theta_2\cos(\theta_1-\theta_2). \quad (32)$$

For solving the Euler-Lagrange equation in general coordinates θ_2, equation (23) is obtained in the same steps, as follows

$$\ddot{x} = l_1\sin(\theta_1-\theta_2)\,\ddot{\theta}_1 + l_1\cos(\theta_1-\theta_2)\,\dot{\theta}_1^{\,2} + (l_2+x)\dot{\theta}_2^{\,2} + g\cos\theta_2 - \frac{kx}{m_2} \quad (33)$$

To fulfill the equation (27), variable $\ddot{x}$ multiplied by $m_2\sin(\theta_1-\theta_2)$, then we obtained the expression

$$m_2\sin(\theta_1-\theta_2)\,\ddot{x} = m_2 l_1\sin^2(\theta_1-\theta_2)\,\ddot{\theta}_1 + m_2 l_1\sin(\theta_1 - \theta_2)\cos(\theta_1-\theta_2)\,\dot{\theta}_1^{\,2} + m_2(l_2+x)\sin(\theta_1 - \theta_2)\,\dot{\theta}_2^{\,2} + m_2 g\cos\theta_2\sin(\theta_1-\theta_2) - kx\sin(\theta_1-\theta_2). \quad (34)$$

Therefore, substitute equation (32) and (34) into equation (27), we obtained

$$m_1 l_1\ddot{\theta}_1 = -kx\sin(\theta_1-\theta_2) - m_1 g\sin\theta_1. \quad (35)$$

To get the equation of motion at the general coordinates θ_2 according to equation (31), we can multiply equation (35) by $-\frac{\cos(\theta_1-\theta_2)}{m_1}$ we obtained

$$-l_1\cos(\theta_1-\theta_2)\,\ddot{\theta}_1 = \frac{kx}{m_1}\sin 2(\theta_1-\theta_2) + g\sin\theta_1\cos(\theta_1-\theta_2). \quad (36)$$

Then, substitute the equation (36) to equation (31)

$$(l_2+x)\ddot{\theta}_2 = \frac{kx}{m_1}\sin 2(\theta_! - \theta_2) + g\sin\theta_1\cos(\theta_1-\theta_2) - 2\dot{\theta}_2\dot{x} + l_1\sin(\theta_1-\theta_2)\,\dot{\theta}_1^{\,2} - g\sin\theta_2. \quad (37)$$

To get the equation of motion at the general coordinates x according to equation (33), we can multiply equation (35) by $\frac{\sin(\theta_1-\theta_2)}{m_1}$

$$l_1\sin(\theta_1-\theta_2)\,\ddot{\theta}_1 = -\frac{kx}{m_1}\sin^2(\theta_1-\theta_2) - g\sin\theta_1\sin(\theta_1-\theta_2). \quad (38)$$

Then, substitute the equation (38) to equation (33)

$$\ddot{x} = l_1 \cos(\theta_1 - \theta_2)\,\dot{\theta_1}^2 + (l_2 + x)\dot{\theta_2}^2 - \frac{kx}{m_1}\sin^2(\theta_1 - \theta_2) - g\sin\theta_1 \sin(\theta_1 - \theta_2) + g\cos\theta_2 - \frac{kx}{m_2}. \tag{39}$$

From equation (35), (37), and (39) we obtained the second order differential equation of generalized coordinates θ_1, θ_2, and x

$$\ddot{\theta_1} = \frac{-kx\sin(\theta_1 - \theta_2) - m_1 g \sin\theta_1}{m_1 l_1} \tag{40}$$

$$\ddot{\theta_1} = \frac{l_1 \sin(\theta_1 - \theta_2)\,\dot{\theta_1}^2 - 2\dot{\theta_2}\dot{x}}{(l_2 + x)} + \frac{\frac{kx}{m_1}\sin 2(\theta_1 - \theta_2) + g\sin\theta_1\cos(\theta_1 - \theta_2) - g\sin\theta_2}{(l_2 + x)} \tag{41}$$

$$\ddot{x} = l_1 \cos(\theta_1 - \theta_2)\,\dot{\theta_1} + (l_2 + x)\dot{\theta_2}^2 - \frac{kx}{m_1}\sin^2(\theta_1 - \theta_2) - g\sin\theta_1\sin(\theta_1 - \theta_2) + g\cos\theta_2 - \frac{kx}{m_2}. \tag{42}$$

The equations of motion in terms of second order differential equations of θ_1, θ_2, and x are obtained. The visualizations of the dynamical motion are presented in this paper. Dynamical behavior of θ_1, θ_2, and x with respect to time t in this case are presented in figure (2), (3), and (4) with parameters $m_1 = 0.1$ kg, $m_2 = 0.1$ kg, $l_1 = 1$ m, $l_2 = 1$ m, $g = 9.8\ m/s^2$, and $k = 100$ N/s. The initial parameter values given are $\theta_1 = \frac{\pi}{3}°$, $\theta_2 = -\pi°$, $x = 0$ m, $\dot{\theta_1} = 1$ rad/s, $\dot{\theta_2} = 1$ rad/s and $\dot{x} = 0$ rad/s. These parameters and initial values are chosen for ease the graphing. It shows the graph in the form of periodic motion with different amplitude and frequency. The amplitude is shown in the form of the peak in each vibration and the frequency is indicated by the intersection of each vibration and the horizontal axis.

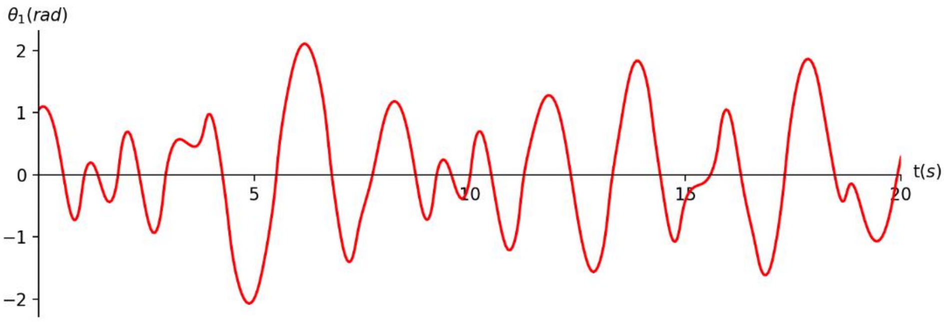

Figure 2. The dynamics of the generalized coordinate θ_1 with respect to time t

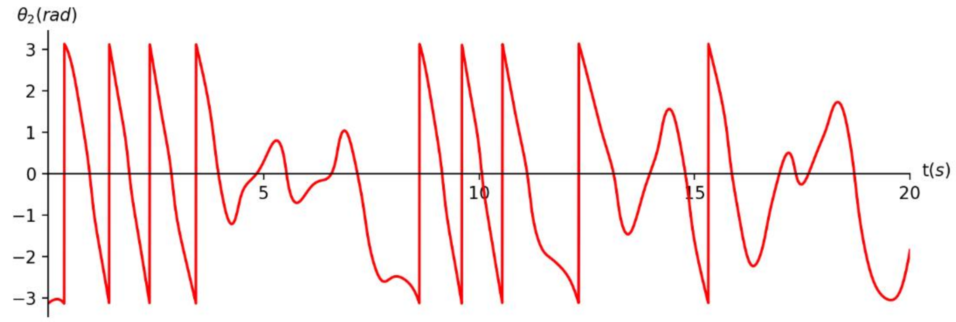

Figure 3. The dynamics of the generalized coordinate θ_2 with respect to time t

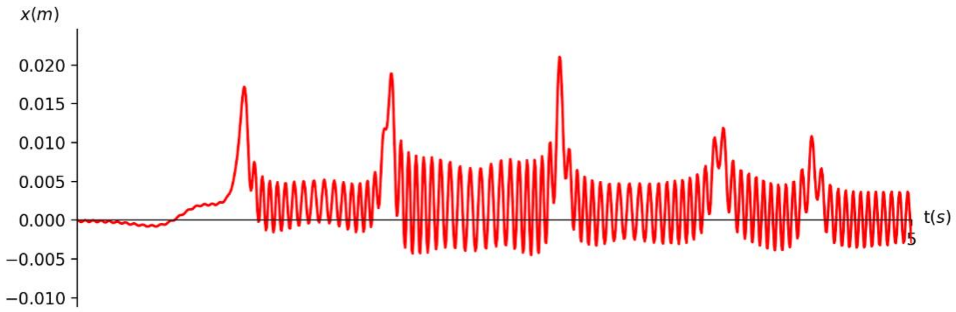

Figure 4. The dynamics of the generalized coordinate x with respect to time t

Figure 2 shows the position θ_1 with respect to time t. It has periodic motion with wavelength varies from one complete cycle to another. The different height indices that the amplitude of the first pendulum varies over the time t. Evolution of the amplitude is unpredicted. These variation looks like Lyapunov exponent in Elbori & Abdalsmd (2017). Figure 3 shows the positions θ_2 with respect to time t. The frequency of the pendulum varies over the time. The amplitude of the second pendulum varies and unpredicted, but the amplitude has maximum limit value. Figure 4 also shows picture was like Figure 2 and 3.

Overall, the figures shows that the motion of the system leads to chaotic behavior, but this prediction must investigate in further research.

CONCLUSION

Equation of motion for the coupled spring-pendulum system with time dependent are derived using Lagrangian mechanics formulation in chosen generalized coordinates. The equation of motion indicated in equation (40), (41), and (42) are obtained. The mathematical analysis in detail is presented. In addition, the dynamical behaviors are described using the visualization graph to make it easier for students to understand. For further investigation, one can analyze the chaotic behavior and Poincare section of this mechanical system.

REFERENCES

Biglari, H., Jami, R.A., (2016). The double pendulum numerical analysis with Lagrangian and the Hamiltonian equations of motions. *Conference paper: International conference on mechanical and aerospace engineering*, pp 1-12.

de la Peña, L., Cetto, A. M., & Valdés-Hernández, A. (2020). Power and beauty of the Lagrange equations. *Revista Mexicana de Física E*, *17*(1 Jan-Jun), 47-54.

Elbori, A., & Abdalsmd, L. (2017). Simulation of double pendulum. *J. Softw. Eng. Simul*, *3*(7), 1-13.

Hamill, P. (2014). *A student's guide to Lagrangians and Hamiltonians*. Cambridge University Press.

Nenuwe, N. O. (2019). Application of Lagrange equations to 2D double spring-pendulum in generalized coordinates. *Ruhuna Journal of Science*, *10*(2).

Rafat, M. Z., Wheatland, M. S., & Bedding, T. R. (2009). Dynamics of a double pendulum with distributed mass. *American Journal of Physics*, *77*(3), 216-223.