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The Study of Kothe's Conjecture for Some Rings

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Abstract

One of the unsolved problems in mathematics especially for the theory of ring and ideal is Kothe's conjecture. It stated whether if a ring R has no nil ideal except $\{0\}$ then R has no one-sided nil ideal except $\{0\}$. This question is simple, but very complicated to be solved. Mathematicians developed some equivalent statements to Kothe's conjecture to simplify this conjecture. Although this conjecture has proven for some rings, but until now it still open for general ring, especially for non-commutative ring. The purpose of this paper is to study about Kothe's conjecture for some rings. Based on literature study and observation, we conclude that Kothe's conjecture is true for commutative ring. In additional results, we state the counterexample of the invers of Kothe's conjecture and study more deeply in some non-commutative rings, those are $Q(\mathbb{R})$ and matrix ring. The result is positive solution for some spesific case of non-commutative rings.

Keywords:

Kothe's conjecture, nil ideal, ring.

1. Introduction

The study of abstract algebra became increasingly developed with the introduction of the concept of ring and ideal. The ideal concept is motivated by the need to define a ring factor. This leds to the need for a concept analogous to the normal subgroup but with two binary operations as defined in the ring. According to Judson & Beezer (2015) and Fraleigh (1988), I is an ideal of ring R when it satisfies following conditions, those are: (1) I is a subring of R; (2) for every a in I, r in R implies ar in I; and (3) for every a in I and r in R implies ra in I. If just one of (2) or (3) is satisfied, then we called I as one-sided ideal. According to Smoktunowicz (2006) and Ferrero (2001), an element a in ring R is said to be nilpotent if there exists natural number n such that $a^n = 0$. Ring R is said to be nil ring if all of this element are nilpotent. The idea of nil ideal is similar to the nil ring definition.

Kothe's conjecture is a basic question related to nil ideal. According to Hadas (1996), Smoktunowicz (2006), and Ferrero (2001), Kothe's conjecture stated that if ring R has no nil ideal except $\{0\}$, then ring R has no one-sided nil ideal except $\{0\}$. Altough this question is simple, but it is not simple to solve. It was proposed about 90 years ago and still open until now.

There are some equivalent sentences to Kothe's conjecture that have published to simplify and make another approach to solve Kothe's conjecture. Some of those sentences are applicable to solve this conjecture for some non-commutative rings, but until now there is not generalized yet, especially in the case of non-commutative rings. Consequently, we still don't understand the general applicability of this conjecture. Since this conjecture is fundamental to our study in abstract algebra, it needs some specific observation in some non-commutative rings. More specific observations not only help not to miss (or even find) counter examples, but also help to generalize and explore special cases related to Kothe's conjecture. We can collect a lot of informations related to Kothe's conjecture and find the connections between them. Based on those arguments, the objectives of this paper are to explain the applicability of Kothe's conjecture for some rings.

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2. Discussion

There are some equivalent sentences to Kothe's conjecture that have established to simplify and make another approach to solve Kothe's conjecture. According to Smoktunowicz (2006) and Ferrero (2001) the following statements are equivalent to Kothe's conjecture: (1) the sum of two right (left) ideals in any ring is nil; (2) for every nil ring R, the ring $M_n(R)$ is nil; (3) for every nil ring R, the polynomial ring $R[x]$ over R is Jacobson radical; (4) every ring which is sum of a nilpotent subring and a nil subring must be nil; and (5) for every nil ring R, the polynomial ring $R[x]$ over R is not left primitive. Based on those statements, we can extend our understanding to Kothe's conjecture and check some rings in several ways.

Before we continue to explore more, we have to state some interesting facts about ideal in some rings. It is well known that the ideal of any field F are just $\{0\}$ and F. This is because of the fact from definition of ideal, that every ideal I of ring R contains unit, then $I = R$. We continue to study more in matrix ring. According to Sands (1956), for every ring R and I is ideal of R, then $M_n(I)$ is ideal of $M_n(R)$. The proof is straightforward from the definition of ideal. Furthermore, he also stated about the maximal and prime ideal in $M_n(R)$. The prime ideal in $M_n(R)$ is set $M_n(A)$ such that A is prime ideal of R. Let A be a maximal ideal in a ring R, then $M_n(A)$ is a maximal ideal in $M_n(R)$ if and only if A is a prime ideal in R.

More informations about ideal in $M_n(R)$ have published by Newman & Pierce (1969), that is if R is principal ideal ring then every ideal of $M_n(R)$ is principal. Furthermore, every ideal in $M_n(R)$ is in the form $M_n(I)$ for some ideal I in R. According to Radjabalipour & Yahaghi (2007), if D is division ring, then this following statements are equivalent, (1) D is a division ring, (2) left ideal of $M_n(D)$ is the set of matrices in $M_n(D)$ which first r columns are arbitrary elements of D and the remaining $n - r$ columns are all zeros, for some non-negative integer $r \leq n$, and (3) right ideal of $M_n(D)$ is the set of matrices in $M_n(D)$ which first r rows is arbitrary elements of D and the remaining $n - r$ rows are all zeros, for some non-negative integer $r \leq n$.

There are some addition results related to Kothe's conjecture:

2.1. Kothe's conjecture is true for commutative ring

Proof:

Let R be commutative ring. We are going to prove the contraposition of Kothe's conjecture, that is if R has one-sided nil ideal except $\{0\}$, then R has nil ideal except $\{0\}$. Without lost of generality, we assume that this ideal is left ideal I_L of R. Since I_L is nil ideal and $I_L \neq \{0\}$, then there exists nilpotent element $a \neq 0$ in I_L. But, since R is commutative ring, then I_L is also right ideal of R. Consequently, I_L is nil ideal of R except $\{0\}$.

It is an obvious fact from the properties of commutative ring, that is if R is a commutative ring then every left ideal of R is also right ideal of R. This fact leads us to the conclusion that Kothe's conjecture is always true for all commutative rings. Obviously, this conjecture is true for every field, integral domain, and Euclidean domain.

2.2. Kothe's conjecture is true for real quaternion

It makes sense to do more observation to non-commutative rings, since we have known that Kothe's conjecture is always true for every commutative ring. One example of non-commutative ring is real quaternion, denoted by $Q(\mathbb{R})$. We stated $Q(\mathbb{R}) = \{a_0 + a_1 i + a_2 j + a_3 k | a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ and has an additive rule, that is $ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j, i^2 = j^2 = k^2 = -1$. We are going to prove that Kothe's conjecture is true for $Q(\mathbb{R})$.

Proof:

Take any x, y in $Q(\mathbb{R})$ which is $xy = 0$. Clear that $|xy| = 0 = |x||y|$. Because $|x|, |y|$ are real numbers and $\mathbb{R}$ is an integral domain, then $|x| = 0$ or $|y| = 0$. Consequently, it must be $x = 0$ or $y = 0$. This fact leads to the conclution that $Q(\mathbb{R})$ has no zero divisor. In other word, for every $a \neq 0$ in $Q(\mathbb{R})$, a non zero element b in $Q(\mathbb{R})$ such that $ab = 0$ is not exists. This statement implies that there is no nilpotent element in $Q(\mathbb{R})$ except 0. Thus, $Q(\mathbb{R})$ does not have one sided nil ideal except $\{0\}$ and $Q(\mathbb{R})$ does not have nil ideal except $\{0\}$. If we want to disprove the truth of Kothe's conjecture in $Q(\mathbb{R})$, then we have to show that there is no nil ideal except $\{0\}$ in $Q(\mathbb{R})$ and there exists one sided nil ideal except $\{0\}$ in $Q(\mathbb{R})$. But it is impossible since the only nilpotent element in $Q(\mathbb{R})$ is 0.

2.3. If D is division ring then Kothe's conjecture is true for $M_n(D)$

I have stated some important results about ideals in matrix ring that I want to use in order to see the applicability of Kothe's conjecture for matrix ring in the beginning of our discussion, those are, (1) every ideal in $M_n(R)$ is in the form $M_n(I)$ for some ideal I in R, (2) for division ring D, left ideal of $M_n(D)$ is the set of matrices in $M_n(D)$ which first r columns are arbitrary elements of D and the remaining $n-r$ columns are all zeros, for some non-negative integer $r \leq n$, and (3) for division ring D, right ideal of $M_n(D)$ is the set of matrices in $M_n(D)$ which first r rows is arbitrary elements of D and the remaining $n-r$ rows are all zeros, for some non-negative integer $r \leq n$.

Proof:

It is clear that every non zero element in D is unit. Consequently, the ideal of D is only $\{0\}$ and D itself. According to the property (1), we conclude that the ideals of $M_n(D)$ are $\{0_n\}$ and $M_n(D)$. Of course $M_n(D)$ is not nil ideal since if we choose A in $M_n(D)$ such that $c_{11} \neq 0$ in A, then $A^n \neq 0$, for all natural number n. Thus, $M_n(D)$ has no other nil ideal except $\{0_n\}$. According to the properties (2) and (3), we are going to set non zero left ideal I_L and right ideal I_R straightforward from those properties. Choose element L in I_L such that $c_{i1} \neq 0$ for $i = 1, 2, \ldots, n$. It is clear that $L^m \neq 0$ for all natural number m. It concludes that there is no left nil ideal in $M_n(D)$ except $\{0_n\}$. Similarly, we choose element R in I_R such that $c_{1i} \neq 0$ for $i = 1$, 2, …, n. It is clear that $R^m \neq 0$ for all natural number m. It concludes that there is no right nil ideal in $M_n(D)$ except $\{0_n\}$. Thus, there is no one sided nil ideal in $M_n(D)$ except $\{0_n\}$. This statement proves the truth of Kothe's conjecture in $M_n(D)$. Consequently, Kothe's conjecture is also true for $M_n(F)$ where F is a field, $M_n\big(Q(\mathbb{R})\big)$, and $M_n(\mathbb{R})$.

2.4. The inverse of Kothe's conjecture is not true

The inverse of Kothe's conjecture stated that if ring R has nil ideal except $\{0\}$, then R has one-sided nil ideal except $\{0\}$. We will prove that this statement is false by giving a very spesific counter example.

Proof:

Setting ring $R = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{R}\right\}$ and $I = \left\{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \middle| x \in \mathbb{R}\right\}$ ideal of R. Take any A in I. Clear that $A = \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix}$ for some $p \in \mathbb{R}$. But, we can see that $A^2 = 0_2$, so A is nilpotent. Clear that R has nil ideal except $\{0_n\}$, that is I. According to the properties (2) and (3) from the previous explanation, it makes sense to define one-sided ideals $K_1 = \left\{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \middle| x, y \in \mathbb{R}\right\}$ and $K_2 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \middle| x, y \in \mathbb{R}\right\}$ are right ideal of R. This is because any ideal of ring must be an ideal of its subring. We consider the fact that R is subring of $M_n(\mathbb{R})$. But it's easy to check that K_1 is also left ideal in R, thus K_1 is ideal of R. Obviously, K_2 is not one-sided nil ideal, since $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is not nilpotent elements. But, it is possible to find other one-sided ideal of R such that not ideal of $M_n(\mathbb{R})$. Since it is the special case of matrix ring, we have to collect all subsets of R such that they have the possibility to become ideal of R.

It is important that we never want to consider the case of $S = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in P, P \subset \mathbb{R}\right\}$ since $\mathbb{R}$ has no other ideal except $\{0\}$ and $\mathbb{R}$ itself, so S never be an ideal of R. Thus we want to consider the remaining subsets of R as follow:

- The subsets contain all of matrices in R that one place in c_{ij} replaced by 0.

  We set the subsets of R by $K_3 = \left\{\begin{pmatrix} x & x \\ 0 & x \end{pmatrix} \middle| x \in \mathbb{R}\right\}$, $K_4 = \left\{\begin{pmatrix} x & y \\ 0 & y \end{pmatrix} \middle| x, y \in \mathbb{R}\right\}$, $K_5 = \left\{\begin{pmatrix} x & x \\ 0 & y \end{pmatrix} \middle| x, y \in \mathbb{R}\right\}$, and $K_6 = \left\{\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \middle| x, y \in \mathbb{R}\right\}$. It is easy to check by take arbitrary element from those sets and multiply by arbitrary element in R from both sides. We can conclude that no one of those sets become one-sided ideal of R since they are not close under the multiplication with some elements in R.
- The subsets contain all of matrices in R that two places in c_{ij} replaced by 0 except K_1.

  We set the subsets of R by $K_7 = \left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| x, y \in \mathbb{R}\right\}$, $K_8 = \left\{\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \middle| x \in \mathbb{R}\right\}$, $K_9 = \left\{\begin{pmatrix} 0 & x \\ 0 & x \end{pmatrix} \middle| x, y \in \mathbb{R}\right\}$, $K_{10} = \left\{\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \middle| x, y \in \mathbb{R}\right\}$, and $K_{11} = \left\{\begin{pmatrix} x & x \\ 0 & 0 \end{pmatrix} \middle| x \in \mathbb{R}\right\}$. It is easy to check by take arbitrary element from those sets and multiply by arbitrary element in R from both sides. We can conclude that K_9 is right ideal of R, K_{10} is ideal of R, and K_{11} is left ideal of R. Observe that K_9 and K_{11} are not one-sided nil ideal since $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ are not nilpotent.

- The subsets contain all of matrices in R that three places in c_{ij} replaced by 0 except K_2 and I.

  We set the subset of R by $K_{12} = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \middle| x \in \mathbb{R} \right\}$. It is easy to check by take arbitrary element from K_{12} and multiply by arbitrary element from R from both sides. We conclude that K_{12} is left ideal of R and it is not one sided nil ideal of R since $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is not nilpotent.

Consequently, there is nil ideal I in R except $\{0\}$, but there is no one sided nil ideal in R except $\{0\}$. Thus, R is a counterexample of the inverse of Kothe's conjecture.

2.5. Kothe's conjecture is true for $M_n(S)$, where S is non zero nil subring of $\mathbb{Z}_c$

In this case, of course we don't want to consider for prime number c, since $\mathbb{Z}_c$ will be an integral domain and there is no nonzero nilpotent element in every integral domain. Thus, we choose nonprime c. Since c is nonprime, then we can collect some zero divisors of $\mathbb{Z}_c$ which are nilpotent. We define S as a collection of all nilpotent element in $\mathbb{Z}_c$. In order to reach our goal, we have to fullfill the following conditions:

- Lemma 1: every product of nilpotent element in commutative ring R is always nilpotent.

  Proof:

  Let p, q are nilpotent elements in R. Clear that there exists natural numbers m, n such that $p^n = 0$ and $q^m = 0$. Choose $r = \min\{m, n\}$. Consequently, we get $(pq)^r = p^r q^r = 0$. Thus, the product of every nilpotent element in $\mathbb{Z}_c$ (commutative ring) is nilpotent.

- Lemma 2: every sum of nilpotent element in commutative ring R is always nilpotent.

  Proof:

  Let p, q are nilpotent elements in R. Clear that there exists natural numbers m, n such that $p^n = 0$ and $q^m = 0$. Choose r such that for $i = 0,1,2, \dots, r$, $r - i \geq \max\{m, n\}$ or $i \geq \max\{m, n\}$. In other word, when $i \approx \left\lfloor \frac{r}{2} \right\rfloor$, $r - i \geq \max\{m, n\}$ or $i \geq \max\{m, n\}$. Then, by using binomial expansion, we see that $(p + q)^r = \sum_{i=0}^{r} \binom{r}{i} p^i q^{r-i} = 0$, since the product of $p^i q^{r-i}$ is always 0 for each i by our consideration. Thus, every sum of nilpotent element in R is nilpotent.

- Lemma 3: every product of nilpotent element and arbitrary element in commutative ring R is always nilpotent.

  Proof:

  Take any element p in S and q in R. Clear that there exists natural numbers n such that $p^n = 0$. Since R is commutative ring, then $(pq)^n = p^n q^n = 0. q^n = 0$. Thus, every product of nilpotent element and arbitrary element in commutative ring R is always nilpotent.

- S is ideal of R.

  Proof:

  We want to prove stronger structure, since every ideal is always subring but not every subring is ideal. Thus, by proving S is ideal of R, we get S is subring of R. By lemma 2, we also deduce that every a, b in S, then $a - b$ always in S. By lemma 3, since $\mathbb{Z}_n$ is commutative ring, then we can deduce that every a in S and r in R, then ar and ra are always in S. Thus, S is ideal of R. Based on those arguments, we can say that S is nil subring of R.

- $M_n(S)$ is nil ring.

  Proof:

  We use the equivalent statements to Kothe's conjecture that is if R is nil ring then $M_n(R)$ is nil. Since Kothe's conjecture is always true for every commutative ring and S is commutative ring, then Kothe's conjecture is true for S. Because S is nil ring and Kothe's conjecture is true for S, then $M_n(S)$ is also nil ring.

- $M_2(M_n(S))$ is nil ring.

  This is the final statement that we have to prove in order to reach our goal. Observe that each element of $M_2(M_n(S))$ is in the form $P = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, where A_1, A_2, A_3, A_4 are in $M_n(S)$, (clear that all of them are nilpotent elements). We are going to claim that there exists some M such that $P^M = 0_2$. We start with identification of S, that is if $c = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_m^{\alpha_m}$, where p_i are primes, then clear that the set $S = \left\{\overline{\prod_{\iota=1}^{m} p_\iota^{\beta_\iota}} \mid 1 \le \beta_i \le \alpha_i\right\}$. Clear that for some natural number t, $P^t = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, with each of B_i is a matrix over S. We know that each B_i is the sum of some matrices in $M_n(S)$. Clear that all entries in B_k are in the form $\overline{\prod_{\iota=1}^{m} p_\iota^{\omega_\iota} K_\iota}$, for some natural number K_i. When we multiply B_k and B_l in $M_n(S)$, then each entry of the product of $B_k B_l$ is in the form $\overline{\prod_{\iota=1}^{m} p_\iota^{\gamma_\iota} U_\iota}$, for some natural number U_i, which $\gamma_i > \omega_i$ for $i = 1,2,3,\dots,m$. Again, when we add B_k and B_l is $M_n(S)$, we add elements c_{ij} in B_k and B_l which is $\overline{\prod_{\iota=1}^{m} p_\iota^{\theta_{k_\iota}} R_{k_\iota}} + \overline{\prod_{\iota=1}^{m} p_\iota^{\theta_{l_\iota}} R_{l_\iota}} = \overline{\prod_{\iota=1}^{m} p_\iota^{\min(\theta_{k_\iota}, \theta_{l_\iota})} Q_\iota}$, for some positive integer Q_i. Then, we choose a great value of M such that each entry of B_i's in P^M is $\overline{\prod_{\iota=1}^{m} p_\iota^{\mu_\iota} V_\iota}$, for some positive integer V_i such that each $\mu_i \ge \alpha_i$, for $i = 1,2,\dots,m$. In this condition, we get $P^M = 0$. Consequently, $M_2(M_n(S))$ is nil ring.

In the end of this explanation, we want to state that there is no guarantee that the product of nilpotent element in non-commutative ring is always nilpotent. But, in this special case it satisfies this property because $M_n(S)$ is nil ring. In general, according to Sullivan (2008) and Wu (1987), when $n \ge 3$, then any singular matrix can be expressed as a product of nilpotent matrices.

3. Conclusion

According to our discussion, we conclude that Kothe's conjecture is true for commutative rings, real quaternion, $M_n(D)$ where D is a division ring, and $M_n(S)$ where S is collection of all nilpotent elements in $\mathbb{Z}_c$ and c is composit natural number. We also state the counterexample of the inverse of Kothe's conjecture by choosing very spesific matrix ring. There are some suggestions to future study about Kothe's conjecture. First, we have to explore more in non-commutative rings. Second, we have to pay more attention if we work in very spesific case, because if I is the only ideal of R, and S is subring of R, then I is possibly not the only ideal of S.

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