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Numerical results and stability of ADI method to two-dimensional advection-diffusion equations with half step of time

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Abstract

We are concerned with the study of stability and numerical results of discretization for alternating direct implicit (ADI) method to two-dimensional advection-diffusion equation. We first discrete two-dimensional advection-diffusion by using forward difference of time and central difference of space. Then, we have two matrices with the step size of time $(n, n+0.5)$ and $(n+0.5, n+1)$, in which this technique is the idea of ADI method. The stability is established by using the Von-Neumann stability technique where the stability characteristic of ADI method is unconditionally stable.

Keywords:
stability, advection-diffusion, alternating direct implicit, finite difference.

1. Introduction

The alternating direct implicit method has been extensively studied by many researchers for some cases of partial differential equations. The characteristic of unconditional stability of this method is one of some reasons that this numerical method is easy to get the convergent results even it needs much time to do discretization to partial differential equations.

The convergence of high order alternating direct implicit to two-dimensional diffusion equation with fractional operator of time has been studied in Cui (2013), where the Caputo time-derivative was approached in L^1 approximations and the second derivative of space was approached by compact finite difference.Douglas (1962) generalized an alternating direct implicit to handle the space variables problem which was a modification of the Crank-Nicolson. Moreover, the boundary of a rectangular parallelepiped was considerable for nonlinear parabolic and elliptic problems.The alternating direct implicit was addressed in multidimensional differential equations which reduced from the multidimensional problem into sequence of one-dimensional problem Douglas (1955, 1964).

Li et al. (2013) introduced alternating direct implicit combined with finite element to two-dimensional reaction-subdiffusion equation with fractional time-derivative, where the stability and error estimate were studied. Peaceman and Rachford (1955) investigated the numerical results to two-dimensional heat equation (which is well known as two-dimensional diffusion equation) by using the standard alternating direct implicit. Wang and Vong (2015) studied the numerical results to fractional time-derivative reaction-subdiffusion equation by employing compact alternating direct implicit finite difference, which means that the original equation was first transformed into equivalent form and then it was discretized by finite difference scheme. The numerical results of fractional sub-diffusion equation employing the new alternating direct implicit and backward Euler method was studied in Zhang and Sun (2011). Moreover, Zhanget al. (2012) addressed the Crank-Nicolson compact alternating direct implicit finite difference scheme, where the stability and convergence were established by energy method.

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Based on the last studies, we present the numerical results to two-dimensional advection-diffusion equation by using the standard alternating direct implicit with half step size of time. We are also interested with the stability criteria of alternating direct implicit to two-dimensional advection-diffusion equation by employing Von-Neumann stability. Moreover, this paper is organized as follows. Section 2, we study the discretization processes of alternating direct implicit to two-dimensional advection-diffusion equation by employing the forward difference in time and central difference in first and second derivate of space. We further decompose the tridiagonal matrices which can be solved implicitly by MATLAB software with various of $\Delta t, \Delta x = \Delta y$, in Section 3. Finally, in Section 4, we investigate the stability of the discretization results by employing Von-Neumann technique and it can be concluded that the alternating direct implicit to two-dimensional advection-diffusion equation is unconditional stable.

2. Methodology

There are some steps to study the numerical results and stability of ADI method to two-dimensional advection-diffusion equation with half step of time, including:

1. Applying the finite difference of ADI method to two-dimensional advection-diffusion with half step size of time to establish the discretization.
2. Grouping the same index of time for discretization results into two tri diagonal matrices.
3. Making the simulation based on the two tri diagonal matrices obtained before by using software MATLAB 2010a.
4. Based on the discretization results, we can derive the stability by employing the Von-Neumann stability technique.

3. ADI scheme to advection-diffusion

We consider the following two-dimensional advection-diffusion equation.

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \tag{1}$$

where the initial condition and boundary conditions are respectively given as follows

$$U(x, y, 0) = e^{-((x^2+y^2)/100)}, \quad (x, y) \in [-20, 20]$$

$$U(-20, y, t) = 0, \quad U(20, y, t) = 0, \quad U(x, -20, t) = 0, \quad U(x, 20, t) = 0$$

We further discrete (1) by applying the following Crank-Nicolson to establish the finite difference.

$$\begin{aligned} \frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t} &= \frac{1}{\Delta x^2}\left(\frac{1}{2}\partial_{xx}^{n+1} + \frac{1}{2}\partial_{xx}^{n}\right) + \frac{1}{\Delta y^2}\left(\frac{1}{2}\partial_{yy}^{n+1} + \frac{1}{2}\partial_{yy}^{n}\right) \\ &\quad - \frac{1}{\Delta x}\left(\frac{1}{2}\partial_{x}^{n+1} + \frac{1}{2}\partial_{x}^{n}\right) - \frac{1}{\Delta y}\left(\frac{1}{2}\partial_{y}^{n+1} + \frac{1}{2}\partial_{y}^{n}\right) \end{aligned} \tag{2}$$

Now, we approach U_t by forward difference, ∂_{xx}, ∂_{yy}, ∂_x, and $\partial_y$by central difference. Moreover, we assume that Δx and Δy are grids for the x-axis and y-axis respectively. Then, (2) becomes

$$\begin{aligned} \frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t} &= \frac{1}{2\Delta x^2}\left(\left[U_{i+1,j}^{n+1} - 2U_{i,j}^{n+1} + U_{i-1,j}^{n+1}\right] + \left[U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i-1,j}^{n}\right]\right) \\ &\quad + \frac{1}{2\Delta y^2}\left(\left[U_{i,j+1}^{n+1} - 2U_{i,j}^{n+1} + U_{i,j-1}^{n+1}\right] + \left[U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n}\right]\right) \\ &\quad - \frac{1}{2\Delta x}\left(\left[U_{i+1,j}^{n+1} - U_{i-1,j}^{n+1}\right] + \left[U_{i+1,j}^{n} - U_{i-1,j}^{n}\right]\right) \\ &\quad - \frac{1}{2\Delta y}\left(\left[U_{i,j+1}^{n+1} - U_{i,j-1}^{n+1}\right] + \left[U_{i,j+1}^{n} - U_{i,j-1}^{n}\right]\right) \\ &= \frac{1}{2}\left(\partial_x^2 + \partial_y^2\right)\left(U_{i,j}^{n+1} + U_{i,j}^{n}\right) - \frac{1}{2}\left(\partial_x + \partial_y\right)\left(U_{i,j}^{n+1} + U_{i,j}^{n}\right) \end{aligned} \tag{3}$$

which gives

$$\left(1-\frac{\Delta t}{2}\left(\partial_x^2+\partial_y^2\right)-\frac{\Delta t}{2}\left(\partial_x+\partial_y\right)\right)U_{i,j}^{n+1}=\left(1+\frac{\Delta t}{2}\left(\partial_x^2+\partial_y^2\right)+\frac{\Delta t}{2}\left(\partial_x+\partial_y\right)\right)U_{i,j}^{n} \quad (4)$$

The idea behind ADI method is to separate two finite difference, the first one is implicit finite difference for derivative of x and the second one is implicit finite difference for derivative of y. Because of this reason, it follows from (3), we have the following two finite difference.

$$\frac{U_{i,j}^{n+\frac{1}{2}}-U_{i,j}^{n}}{\frac{\Delta t}{2}}=\left((\partial_x^2+\partial_x)U_{i,j}^{n+\frac{1}{2}}+\left(\partial_y^2+\partial_y\right)U_{i,j}^{n}\right) \quad (5)$$

and

$$\frac{U_{i,j}^{n+1}-U_{i,j}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}}=\left((\partial_x^2+\partial_x)U_{i,j}^{n+\frac{1}{2}}+\left(\partial_y^2+\partial_y\right)U_{i,j}^{n+1}\right) \quad (6)$$

It follows from (5) and (6), one respectively has

$$\left(1-\frac{\Delta t}{2}(\partial_x^2+\partial_x)\right)U_{i,j}^{n+\frac{1}{2}}=\left(1+\frac{\Delta t}{2}\left(\partial_y^2+\partial_y\right)\right)U_{i,j}^{n} \quad (7)$$

and

$$\left(1-\frac{\Delta t}{2}\left(\partial_y^2+\partial_y\right)\right)U_{i,j}^{n+1}=\left(1+\frac{\Delta t}{2}(\partial_x^2+\partial_x)\right)U_{i,j}^{n+\frac{1}{2}} \quad (8)$$

Combining (7) and (8) gives

$$\begin{aligned}&\left(1-\frac{\Delta t}{2}(\partial_x^2+\partial_x)\right)\left(1-\frac{\Delta t}{2}\left(\partial_y^2+\partial_y\right)\right)U_{i,j}^{n+1}\\&=\left(1+\frac{\Delta t}{2}(\partial_x^2+\partial_x)\right)\left(1+\frac{\Delta t}{2}\left(\partial_y^2+\partial_y\right)\right)U_{i,j}^{n}\end{aligned} \quad (9)$$

4. Numerical results

It follows from (7) and (8), respectively obtained

$$\begin{aligned}&-(s-1)U_{i-1,j}^{n+\frac{1}{2}}+(1+2s)U_{i,j}^{n+\frac{1}{2}}-(s+1)U_{i+1,j}^{n+\frac{1}{2}}\\&=(t+1)U_{i,j-1}^{n}+(1-2t)U_{i,j}^{n}+(t-1)U_{i,j+1}^{n}\end{aligned} \quad (10)$$

and

$$\begin{aligned}&-(t-1)U_{i,j-1}^{n+1}+(1+2t)U_{i,j}^{n+1}-(t+1)U_{i,j+1}^{n+1}\\&=(s+1)U_{i-1,j}^{n+\frac{1}{2}}+(1-2s)U_{i,j}^{n+\frac{1}{2}}+(s-1)U_{i+,j}^{n+\frac{1}{2}}\end{aligned} \quad (11)$$

where $s=\frac{\Delta t}{2\Delta x^2}+\frac{\Delta t}{2\Delta x}$ and $t=\frac{\Delta t}{2\Delta y^2}+\frac{\Delta t}{2\Delta y}$.

We further iterate (10) for $i=1,2,\dots,N$ and $j=1,2,\dots,N$ and also substitute the boundary conditions to (10), one has

$$
\begin{bmatrix}
1+2s & -(s-1) & 0 & \cdots & \cdots & 0 \\
-(s+1) & 1+2s & 0 & \cdots & \cdots & 0 \\
0 & -(s+1) & -(s-1) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\
0 & 0 & 0 & 1+2s & -(s-1) & 0 \\
0 & 0 & 0 & -(s+1) & 1+2s & -(s-1) \\
0 & 0 & 0 & 0 & -(s+1) & 1+2s
\end{bmatrix}
\begin{bmatrix}
U_{1,j}^{n+\frac{1}{2}} \\ U_{2,j}^{n+\frac{1}{2}} \\ U_{3,j}^{n+\frac{1}{2}} \\ \vdots \\ \vdots \\ U_{N-1,j}^{n+\frac{1}{2}} \\ U_{N,j}^{n+\frac{1}{2}}
\end{bmatrix}
=
\begin{bmatrix}
1-2t & t-1 & 0 & \cdots & \cdots & 0 \\
t+1 & 1-2t & 0 & \cdots & \cdots & 0 \\
0 & t+1 & t-1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\
0 & 0 & 0 & 1-2t & t-1 & 0 \\
0 & 0 & 0 & t+1 & 1-2t & t-1 \\
0 & 0 & 0 & 0 & t+1 & 1-2t
\end{bmatrix}
\begin{bmatrix}
U_{1,j}^{n} \\ U_{2,j}^{n} \\ U_{3,j}^{n} \\ \vdots \\ \vdots \\ U_{N-1,j}^{n} \\ U_{N,j}^{n}
\end{bmatrix}
\qquad (12)
$$

Similarly for (11), obtained

$$
\begin{bmatrix}
1+2t & -(t-1) & 0 & \cdots & \cdots & 0 \\
-(t+1) & 1+2t & 0 & \cdots & \cdots & 0 \\
0 & -(t+1) & -(t-1) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\
0 & 0 & 0 & 1+2t & -(t-1) & 0 \\
0 & 0 & 0 & -(t+1) & 1+2t & -(t-1) \\
0 & 0 & 0 & 0 & -(t+1) & 1+2t
\end{bmatrix}
\begin{bmatrix}
U_{1,j}^{n+1} \\ U_{2,j}^{n+1} \\ U_{3,j}^{n+1} \\ \vdots \\ \vdots \\ U_{N-1,j}^{n+1} \\ U_{N,j}^{n+1}
\end{bmatrix}
=
\begin{bmatrix}
1-2s & s-1 & 0 & \cdots & \cdots & 0 \\
s+1 & 1-2s & 0 & \cdots & \cdots & 0 \\
0 & s+1 & s-1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\
0 & 0 & 0 & 1-2s & s-1 & 0 \\
0 & 0 & 0 & s+1 & 1-2s & s-1 \\
0 & 0 & 0 & 0 & s+1 & 1-2s
\end{bmatrix}
\begin{bmatrix}
U_{1,j}^{n+\frac{1}{2}} \\ U_{2,j}^{n+\frac{1}{2}} \\ U_{3,j}^{n+\frac{1}{2}} \\ \vdots \\ \vdots \\ U_{N-1,j}^{n+\frac{1}{2}} \\ U_{N,j}^{n+\frac{1}{2}}
\end{bmatrix}
\qquad (13)
$$

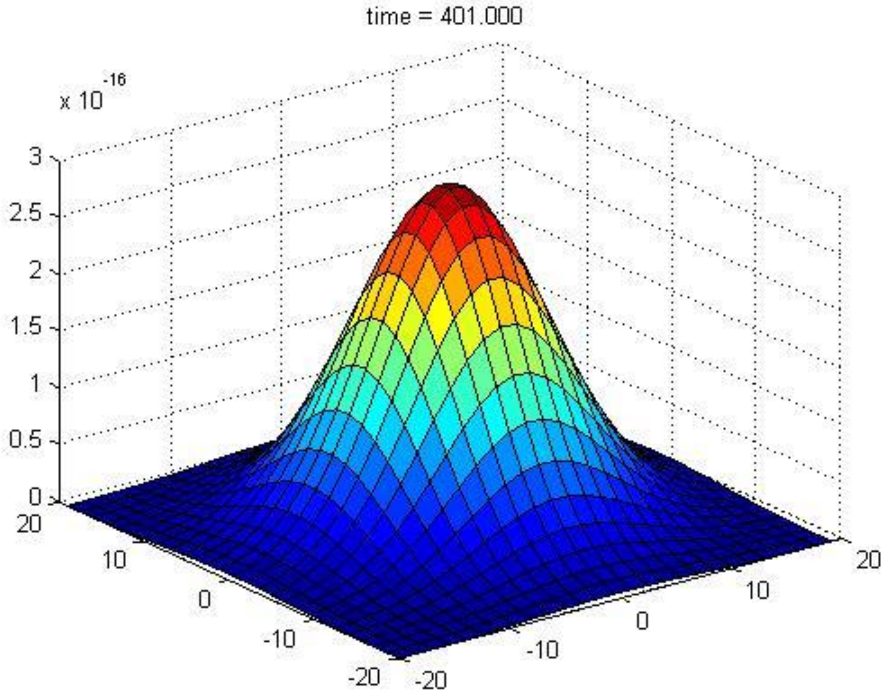

Figure 1. Numerical results of 2D advection-diffusion equation with $\Delta t = 1, \Delta x = \Delta y = 1.5$

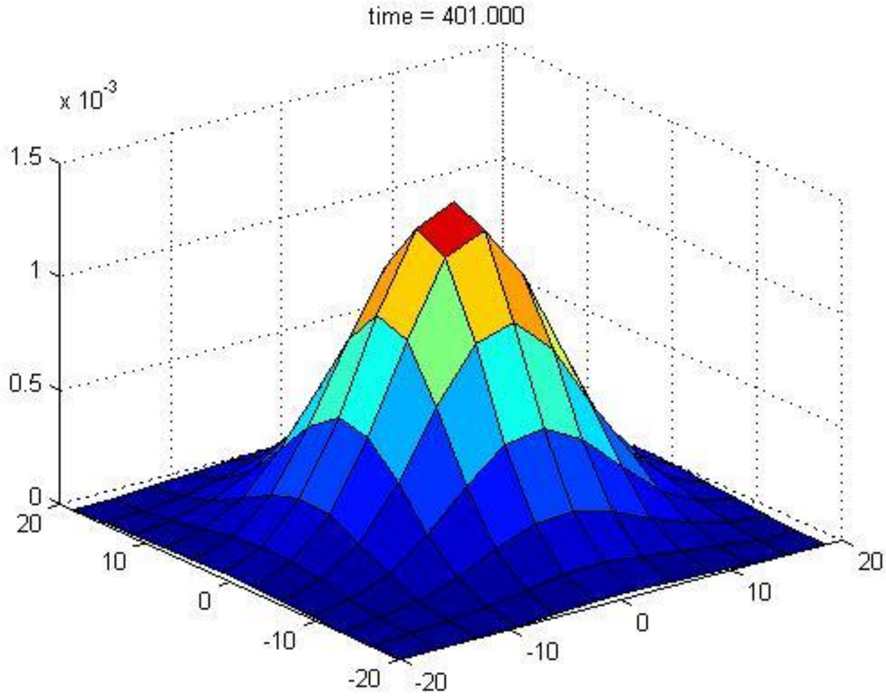

Figure 2. Numerical results of 2D advection-diffusion equation with $\Delta t = 1, \Delta x = \Delta y = 3.5$

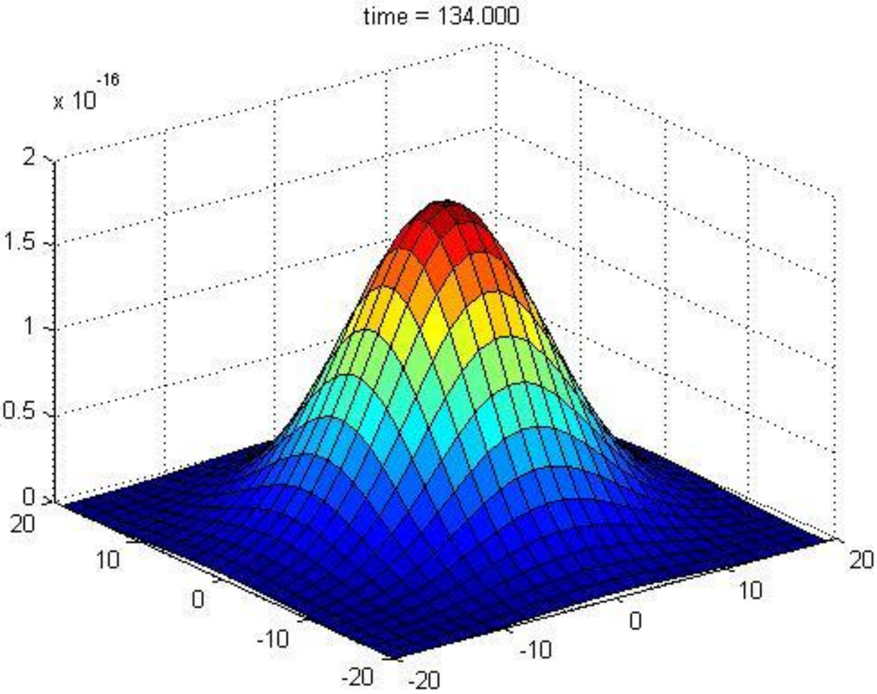

Figure 3. Numerical results of 2D advection-diffusion equation with $\Delta t = 3, \Delta x = \Delta y = 1.5$

Based on the results of numerical simulations on the two-dimensional advection-diffusion equation by applying the ADI numerical method, we can conclude that the ADI numerical method is unconditionally stable. This can be shown from Figure 1 to Figure 3 for various values of spaceΔx, Δy, and time Δt, assuming uniform grid-x and grid-y are the same ($\Delta x = \Delta y$), the graph movement always slopes. In Figure 1 reaches the final value $U(x, y, t) = x\ 10\text{^}(-16)$ with an iteration time of 401 until conditions converge, Figure 2 reaches the value $U(x, y, t) = x\ 10\text{^}(-3)$ with an iteration time of 401 until the conditions converge, and Figure 3 reaches the value of $U(x, y, t) = x\ 10\text{^}(-16)$ with an iteration time of 134 until the conditions converge.

5. Stability of ADI scheme toadvection-diffusion

A numerical scheme is called stable if the error is not increased for all the iteration of time. We further employ the following steps to check whether the ADI method to two-dimensional advection-diffusion equation stable or unstable.

We first substitute $U_{i,j}^n = \lambda^n e^{i\beta_1 ij}$ to (10), one has

$$\begin{aligned} -(s-1)\lambda^{n+\frac{1}{2}}e^{i\beta_1(i-1)j} \quad &+(1+2s)\lambda^{n+\frac{1}{2}}e^{i\beta_1 ij} - (s+1)\lambda^{n+\frac{1}{2}}e^{i\beta_1(i+1)j} \\ &= (t-1)\lambda^n e^{i\beta_1 i(j-1)} + (1-2t)\lambda^n e^{i\beta_1 ij} + (t+1)\lambda^n e^{i\beta_1 i(j+1)} \end{aligned} \tag{14}$$

Dividing both sides by $\lambda^n e^{i\beta_1 ij}$, we have

$$\begin{aligned}
&-(s-1)\lambda^{\frac{1}{2}}e^{i\beta_1 j}+(1+2s)\lambda^{\frac{1}{2}}-(s+1)\lambda^{\frac{1}{2}}e^{i\beta_1 j}=(t-1)e^{-i\beta_1 i}+(1-2t)+(t+1)e^{i\beta_1 i}\\
&\Leftrightarrow -s\lambda^{\frac{1}{2}}\left(e^{-i\beta_1 j}+e^{i\beta_1 j}\right)+\lambda^{\frac{1}{2}}\left(e^{-i\beta_1 j}-e^{i\beta_1 j}\right)+(1+2s)\lambda^{\frac{1}{2}}\\
&\quad =t\left(e^{-i\beta_1 i}+e^{i\beta_1 i}\right)-\left(e^{-i\beta_1 i}-e^{i\beta_1 i}\right)+(1-2t)\\
&\Leftrightarrow -s\lambda^{\frac{1}{2}}(2\cos\beta_1 j)-\lambda^{\frac{1}{2}}(2i\sin\beta_1 j)+(1+2s)\lambda^{\frac{1}{2}}=t(2\cos\beta_1 i)-(2i\cos\beta_1 i)+(1-2t) \qquad (15)\\
&\Leftrightarrow \lambda^{\frac{1}{2}}(-2s\cos\beta_1 j+2s-2i\sin\beta_1 j)=2t\cos\beta_1 i+1-2t-2i\cos\beta_1 i\\
&\Leftrightarrow \lambda_1=\left(\frac{1-4t\sin^2\frac{\beta_1 i}{2}-2i\sin\beta_1 j}{1+4s\sin^2\frac{\beta_1 j}{2}-2i\cos\beta_1 i}\right)^2
\end{aligned}$$

Similarly, we substitute $U_{i,j}^{n}=\lambda^{n}e^{i\beta_2 ij}$ to (11), one has

$$\begin{aligned}
&-(t-1)\lambda^{n+1}e^{i\beta_2 i(j-1)}+(1+2t)\lambda^{n+1}e^{i\beta_2 ij}-(t+1)\lambda^{n+1}e^{i\beta_2 i(j+1)}\\
&=(s-1)\lambda^{n+\frac{1}{2}}e^{i\beta_2(i-1)j}+(1-2s)\lambda^{n+\frac{1}{2}}e^{i\beta_2 ij}+(s+1)\lambda^{n+\frac{1}{2}}e^{i\beta_2(i+1)j} \qquad (16)
\end{aligned}$$

Then, we divide both sides by $\lambda^{n}e^{i\beta_2 ij}$,

$$\begin{aligned}
&-(t-1)\lambda^{\frac{1}{2}}e^{-i\beta_2 i}+(1+2t)\lambda^{\frac{1}{2}}-(t+1)\lambda^{\frac{1}{2}}e^{i\beta_2 i}=(s-1)e^{-i\beta_2 j}+(1-2s)+(s+1)e^{i\beta_2 j}\\
&\Leftrightarrow -t\lambda^{\frac{1}{2}}\left(e^{-i\beta_2 i}+e^{i\beta_2 i}\right)+\lambda^{\frac{1}{2}}\left(e^{-i\beta_2 i}-e^{i\beta_2 i}\right)+(1+2t)\lambda^{\frac{1}{2}}\\
&\quad =s\left(e^{-i\beta_2 j}+e^{i\beta_2 j}\right)-\left(e^{-i\beta_2 j}-e^{i\beta_2 j}\right)+(1-2s)\\
&\Leftrightarrow -t\lambda^{\frac{1}{2}}(2\cos\beta_2 i)-\lambda^{\frac{1}{2}}(2i\sin\beta_2 i)+(1+2t)\lambda^{\frac{1}{2}}=s(2\cos\beta_2 j)-2i\cos\beta_2 j+(1-2s) \qquad (17)\\
&\Leftrightarrow \lambda^{\frac{1}{2}}(-2t\cos\beta_2 i+1+2t-2i\sin\beta_2 i)=2s\cos\beta_2 j-2i\cos\beta_2 j+1-2s\\
&\Leftrightarrow \lambda_2=\left(\frac{1-4s\sin^2\frac{\beta_2 j}{2}-2i\sin\beta_2 i}{1+4t\sin^2\frac{\beta_2 i}{2}-2i\cos\beta_2 j}\right)^2
\end{aligned}$$

Based on the calculations of λ_1, and λ_2, we have non-negative of λ_1, λ_2, and $|(\lambda_1,\lambda_2)|\leq 1$. Then, we can conclude that the ADI scheme for two-dimensional advection-diffusion equation is unconditionally stable.

6. Conclusion

Based on the results and discussion, we can conclude that the ADI method with half step of time gives the stable numerical results for any increment of space and time. This numerical method of ADI has the characteristics of unconditionally stable under the results of Von-Neumann technique.

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