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The influence of Allee effect, refugia, and alternative food on a three-species predator-prey model

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Abstract

This research presents a mathematical model for a three-species predator-prey system, incorporating the Allee effect, refugia, and alternative food. The interactions between prey and intermediate predator, as well as intermediate predator and top predator, are modeled using Holling type II response functions. The resulting system of nonlinear equations yields four equilibrium points, one unstable and three stable locally. Analytical calculations indicate that refugia and alternative food minimally affect the top predator's population growth, while the Allee effect influences the growth of prey and intermediate predator populations. Numerical simulations further support these findings, highlighting the nuanced impacts of these factors on the dynamics of the three-species system.

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1. Introduction

Mathematical modeling is a branch of mathematics that is used as a tool to find solutions to real problems. The real problems faced are sometimes so complex that they are difficult to represent. Therefore, it is necessary to formulate the problem into a simpler mathematical model. Interpretation in the real problem being modeled can be done after solving the mathematical model analytically or numerically. The use of mathematical modeling is closely related to other sciences, for example the field of ecology.

Community ecology is one of the scopes of ecology that examines how interspecies interactions such as predation and competition, can affect community structure and organization. Predation is the relationship between predators and prey that interact due to the need for resources. The balance of the surrounding environment depends on the interaction between predators and prey. For example, the ratio of predator and prey populations in the ecosystem must be considered so as not to cause scarcity or imbalance in the ecosystem. Predator-prey modeling was first introduced by Lotka in 1925 and Volterra in 1926. Therefore, this model is also called the Lotka-Volterra model. This simplest predator-prey model assumes that the population experiences exponential growth and decay.

Along with the development of science, research on predator-prey mathematical modeling is increasingly complex. The population in the ecosystem also decreases and increases unstably over time. In addition, the interaction between predator and prey is not only limited to two species. There are other species that contribute to the balance of the ecosystem. Therefore, many researchers have conducted research on three-species predator-prey interactions and are still being developed today.

Pal *et al.* (2017) have conducted research on the tri-trophic food chain model in the presence of constant immigration in the intermediate predator population. The results show that with constant immigration, the intermediate predator population can maintain its population abundance thus helping the system balance the ecosystem. The three-species predator-prey model has also been studied by Savitri & Panigoro (2020), with each species having a different response function. Ibrahim *et al.* (2021) examined the impact of disease on the predator-prey model with the prey population divided into two sub-populations, namely disease-susceptible prey, and disease-infected prey. In addition, Panja *et al.* (2022) have also examined the competition of two species in a three-species predator-prey model.

The chaos that occurs in food chain interactions can be controlled by incorporating reasonable biological phenomena. Allee effect is one of the biological factors to maintain ecosystem balance introduced by Allee (1931). In population dynamics, the Allee effect refers to a process that affects growth rates for low-density populations. The Allee effect is a biological phenomenon characterized by a correlation between population size or density and the average fitness of individuals of a species that occurs in small population densities. This Allee effect can occur in both predators and prey. Mandal *et al.* (2021) mentioned environmental factors such as difficulty in finding a mate, low probability of mating success, declining inbreeding rates, anti-predator aggression, and predator avoidance due to evolutionary changes as causes of the Allee effect.

A study of mathematical models in the presence of the Allee effect has been reviewed by Parshad *et al.* (2016). The study discussed a food chain model with a strong Allee effect on the top predator and produced a model that showed that overexploitation should be driven by the middle predator. Guin *et al.* (2022) also conducted research on the tri-trophic food chain chaos model by adding the Allee effect to the model. With the Allee effect in the prey and intraspecific competition in the two predators, the results show that both have an important role in controlling chaos in the dynamical system.

Another biological factor that plays a role in system dynamics is refugia. Refugia are places of refuge for isolated populations to survive. The causes of this isolation include climate change, geography, predation risk, or human activities that disrupt ecosystems. In Mukherjee & Maji (2020), mentioned that refugia have two roles. The first is to affect prey growth positively and cause predator growth to be negative because the predation success rate decreases so that prey mortality decreases. The second role is that the impact of refugia may one day be beneficial or even harmful to the population involved.

Prey populations with total refugia can result in the extinction of predator populations. However, if the prey population is not given refugia, it can increase the predator population, thus increasing predation. If there is extinction in the prey population, the predator population will also experience extinction. Therefore, it should be noted that refugia must be adjusted to the conditions and level of predation in the ecosystem (Das & Samanta, 2020). Similarly, research conducted by Mahapatra *et al.*, (2021), refugia on prey should be proportional to the density of predators because it was found that refugia was beneficial for prey and detrimental to predators. Other factors are needed as additional controls to keep the ecosystem in balance, such as the fear effect (Ma *et al.*, 2020), harvesting (Firdiansyah & Nurhidayati, 2021), or the Allee effect (Molla *et al.*, 2022).

Providing alternative food to predator populations is one of the efforts to maintain ecosystem balance because it can reduce the risk of extinction in predator populations. Huda *et al.* (2017) have conducted dynamic analysis research on the Hasting-Powell food chain model with alternative food. Alternative food in this model is only given to the top predator population because it is assumed that the growth of the top predator population only depends on the intermediate predator population. The result of the study is that the provision of alternative food with the appropriate amount has a positive impact on the top predator population. Alternative food plays an important role in ecosystem balance when prey is the only food for predators. Like the research conducted by Das & Samanta (2020), alternative foods inhibit extinction in predators when refugia are independent of predation rates.

In this research, we will study the food chain model compiled by Nath *et al.* (2022) which examines the three-species food chain with refugia and Allee effect in the prey population. Modifications will be made to the model by adding alternative food sources to the intermediate predator and top predator populations, because if the refugia and Allee effect in the prey population are not controlled, the longer it will cause the extinction of the two predator populations.

2. Method

There are stages that must be carried out so that the research objectives can be achieved, which are as follows.

(1) Literature Study. This step is carried out by tracing and studying books or other sources related to the Predator Prey model.

(2) Predator Prey model construction
- Determination of model assumptions, where this stage is the stage of determining model assumptions. Assumptions are needed as a limitation on the extent to which the model will be made.
- Form a predator prey model based on predetermined assumptions. The model formed is a mathematical representation of the basic assumptions before the model is formed.

(3) Analysis and Troubleshooting
- Determine the equilibrium point. The point $x_0 \epsilon R^n$ is called the equilibrium point of the system $\dot{x} = f(x)$ if $f(x_0) = 0$.
- Determining the Jacobian matrix. The general form of the Jacobian Matrix is:
$$J(x,y) = \begin{bmatrix} \frac{\partial f(x,y)}{\partial x} & \frac{\partial f(x,y)}{\partial y} \\ \frac{\partial g(x,y)}{\partial x} & \frac{\partial g(x,y)}{\partial y} \end{bmatrix} \text{ with } \frac{dx}{dt} = f(x,y),\ \frac{dy}{dt} = g(x,y).$$
- Determining eigenvalues. Suppose given a matrix A of size $n \times n$ and a system of homogeneous differential equations of the form $Ax = x,\ x(0) = x_0,\ x \in \mathbb{R}^n$. A nonzero vector $x \in \mathbb{R}^n$ is called an eigenvector of A if for a scalar λ holds $Ax = \lambda x$, *the* scalar value λ is called the eigenvalue of A and x is said to be the eigenvector corresponding to λ.
- Analyze the stability of the equilibrium point. The equilibrium point $\bar{x} \in \mathbb{R}^n$ of the system $\dot{x} = f(x)$ said:
  (a) Stable, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every solution of the system $x = x(t)$ satisfies $\|x(0) - \bar{x}\| < \delta$ then it holds $\|x(t) - \bar{x}\| < \varepsilon$ for every $t \geq 0$,
  (b) Asymptotically stable, if the equilibrium point is $\bar{x} \in \mathbb{R}^n$ is stable and there exists δ_0 where $0 < \delta_0 < \delta$ such that for every solution $x(t)$ which satisfies $\|x(t) - \bar{x}\| < \delta_0$ holds $\lim_{t\to\infty} x(t) = \bar{x}$,
  (c) Unstable, if the equilibrium point $\bar{x} \in \mathbb{R}^n$ does not fulfill criterion (a).

(4) Model Simulation. Simulation of the model is done numerically using the Maple 18 program. This simulation is made to see if the analysis of the equilibrium point of the model is in accordance with the model simulation.

(5) Drawing Conclusions. The last step taken in this research method is drawing conclusions which are the results and discussion of the problem-solving steps that have been carried out previously in accordance with the objectives to be achieved.

3. Results and discussions

3.1. Model Formulation

The growth of prey populations follows logistic growth if there is no interaction between prey and predators. The prey growth rate is influenced by carrying capacity of $\boldsymbol{K_0}$, the natural growth rate of $\boldsymbol{\alpha}$, and the Allee effect *of* $\boldsymbol{\theta > 0}$ which causes the natural mortality rate of the prey population to increase. In addition, the prey growth rate is also influenced by refugia which is denoted by $\boldsymbol{m}$. If some of the refugia in the prey population can protect the prey from predation, then there will be the remaining part, namely $(\boldsymbol{1 - m})$ that does not escape predation and interacts with intermediate predators. If $\boldsymbol{K_0}, \boldsymbol{\alpha} \in \mathbb{R}^+$, $\boldsymbol{C_1}$ is the conversion rate of *prey* consumption to intermediate pr*edator* birth, while $\boldsymbol{A_1}$ and $\boldsymbol{B_1}$ are the maximum predation rate and half-saturation constant for the intermediate predator population then the prey growth rate is

$$\frac{dX}{dT} = \alpha X\left(1 - \frac{X}{K_0}\right)\left(\frac{X}{X+\theta}\right) - \frac{C_1A_1(1-m)XY}{B_1+(1-m)X}. \quad (1)$$

The population growth rate of intermediate predators increases with the predation of prey by intermediate predators and the remaining part of the refugia by prey which eventually interacts with intermediate predators, namely $(\boldsymbol{1-m})$. In addition, the intermediate predator population increases due to the presence of alternative food. The alternative food provided in this model is one type of food and can be consumed by intermediate predators and top predators. As a result, the intermediate predator population will lose out in obtaining alternative food and only get the remaining part of the top predator population by the amount of the top predator population $(\boldsymbol{1-\rho})\boldsymbol{A}$. The number of intermediate predator populations will decrease due to predation by top predators on the intermediate predator population which follows the Holling type II response function, as well as the natural death of the intermediate predator population of $\boldsymbol{D_1}$. The intermediate predator population growth rate model can be expressed as follows.

$$\frac{dY}{dT} = \frac{A_1(1-m)XY}{B_1+(1-m)X} + (1-\rho)AY - \frac{A_2YZ}{B_2+Y} - D_1Y \quad (2)$$

Predation by top predators on intermediate predators causes the population growth rate of top predators to increase. The consumption rate of top predators on intermediate predators follows a Holling type II response function. In addition to predation on intermediate predators, the top predator population also receives a constant amount of alternative food. It is assumed that the top predator population gets alternative food as much as $\boldsymbol{\rho A}$ where $\boldsymbol{0 \le A \le 1}$. Then the number of top predator population decreases due to the natural death of top predator population denoted by $\boldsymbol{D_2}$. If $\boldsymbol{C_2}$ is the conversion rate of intermediate predator consumption to top predator birth, $\boldsymbol{A_2}$ is the maximum predation rate of the top predator population, and $\boldsymbol{B_2}$ is the half-saturation constant for the top predator population, then the model of the top predator population growth rate over time is as follows.

$$\frac{dZ}{dT} = \frac{C_2A_2YZ}{B_2+Y} + \rho AZ - D_2Z \quad (3)$$

Based on equations (1), (2), and (3), the three-species predator-prey model with refugia and Allee effect on prey and provision of alternative food sources to predators takes the form of a nonlinear differential equation as follows.

$$\begin{aligned}
\frac{dX}{dT} &= \alpha X\left(1 - \frac{X}{K_0}\right)\left(\frac{X}{X+\theta}\right) - \frac{C_1A_1(1-m)XY}{B_1+(1-m)X} \\
\frac{dY}{dT} &= \frac{A_1(1-m)XY}{B_1+(1-m)X} + (1-\rho)AY - \frac{A_2YZ}{B_2+Y} - D_1Y \\
\frac{dZ}{dT} &= \frac{C_2A_2YZ}{B_2+Y} + \rho AZ - D_2Z
\end{aligned} \quad (4)$$

The system of equations (4) can be simplified by scaling the system parameters using the following equations.

$$x = \frac{X}{K_0}, \qquad y = \frac{C_1Y}{K_0},\ z = \frac{C_1Z}{C_2K_0}, \qquad \text{and} \qquad t = \alpha T$$

Then the system of equations (4) becomes

$$\begin{aligned}
\frac{dx}{dt} &= x(1-x)\left(\frac{x}{x+\theta}\right) - \frac{a_1(1-m)xy}{1+b_1(1-m)x} \\
\frac{dy}{dt} &= \frac{a_1(1-m)xy}{1+b_1(1-m)x} + (1-\rho)Ay - \frac{a_2yz}{1+b_2y} - d_1y \\
\frac{dz}{dt} &= \frac{a_2yz}{1+b_2y} + \rho Az - d_2z
\end{aligned} \quad (5)$$

$$a_1 = \frac{A_1K_0}{B_1\alpha},\ b_1 = \frac{K_0}{B_1},\ a_2 = \frac{A_2C_2K_0}{B_2C_1\alpha},\ b_2 = \frac{K_0}{B_1C_1},\ d_1\frac{D_1}{\alpha},$$

$$d_2 = \frac{D_2}{\alpha}, \text{ and } \quad \theta = \frac{\theta}{\alpha},$$

with initial values $\boldsymbol{x(0) \ge 0}$, $\boldsymbol{y(0) \ge 0}$, and $\boldsymbol{z(0) \ge 0}$.

3.2. Model Equilibrium Points

The equilibrium point of the model is obtained when the system is in an equilibrium state. A condition when the change in the number of individuals of each population over time is zero is referred to as an equilibrium state. Therefore, the equilibrium point of the model is obtained when the system $\frac{dx}{dt}=\frac{dy}{dt}=\frac{dz}{dt}=\mathbf{0}$.

The system of equations (5) obtained four equilibrium points that exist, namely:

(1) When $x=0$, we get $E_0=(x,y,z)=(0,0,0)$. This is the state when there is no prey in the system, so there will be no intermediate predator and top predator in the system.

(2) When $y=0$, it is obtained $E_1=(x,0,0)=(1,0,0)$. It is a condition that if the intermediate predator does not exist in the system, the top predator will not be able to live. So there will be only prey population in the system.

(3) When $z=0$, it is obtained

$$E_2=(\bar{x},\bar{y},0)=\left(\frac{d_1-(1-\rho)A}{(1-m)[a_1+(1-\rho)Ab_1+d_1b_1]},\frac{(x-x^2)(1+b_1(1-m)x)}{a_1(1-m)(x+\theta)},0\right)$$

with the condition that $d_1>(1-\rho)A$. This is the state when the top predator is not present in the system, so there are only two species that coexist, namely the prey and the intermediate predator.

(4) When $x\neq 0, y\neq 0$, and $z\neq 0$, will be obtained $E_3=(\dot{x},\dot{y},\dot{z})$ where

$\big((1-m)[b_2d_2b_1-Ab_2b_1\rho-a_2b_1]\big)\dot{x}^3+(b_2d_2-Ab_2\rho-a_2+(1-m)[b_2d_2b_1-Ab_2b_1\rho-a_2b_1])\dot{x}^2-\big(b_2d_2-Ab_2\rho-a_2-a_1d_2+Aa_1\rho(1-m)\big)\dot{x}-(1-m)[-a_1d_2\theta+Aa_1\rho]=0,$

$$\dot{y}=\frac{d_2-\rho A}{a_2-d_2b_2+\rho Ab_2},$$

$$\dot{z}=(1+b_2\dot{y})\left[\frac{a_1a_2(1-m)\dot{x}}{1+b_1(1-m)\dot{x}}+\frac{(1-\rho)A-d_1}{a_2}\right].$$

Point E_3 exists if $d_2>\rho A$ and $(1-\rho)A>d_1$. This is the state where there are three species interacting and coexisting in the system, namely prey, intermediate predator, and top predator species.

Theorem 1

a) If $x=0$, then the equilibrium point is obtained $(0,0,0)$
b) If $y=0$, then the equilibrium point is obtained $(1,0,0)$
c) If $z=0$, then an equilibrium point is obtained $\left(\frac{d_1-(1-\rho)A}{(1-m)[a_1+(1-\rho)Ab_1+d_1b_1]},\frac{(x-x^2)(1+b_1(1-m)x)}{a_1(1-m)(x+\theta)},0\right)$ with the condition $d_1>(1-\rho)A$
d) If $x\neq 0, y\neq 0, z\neq 0$, then the equilibrium point is obtained
$\big((1-m)[b_2d_2b_1-Ab_2b_1\rho-a_2b_1]\big)\dot{x}^3+(b_2d_2-Ab_2\rho-a_2+(1-m)[b_2d_2b_1-Ab_2b_1\rho-a_2b_1])\dot{x}^2-\big(b_2d_2-Ab_2\rho-a_2-a_1d_2+Aa_1\rho(1-m)\big)\dot{x}-(1-m)[-a_1d_2\theta+Aa_1\rho]=0,$
$\dot{y}=\frac{d_2-\rho A}{a_2-d_2b_2+\rho Ab_2}$, $\dot{z}=(1+b_2\dot{y})\left[\frac{a_1a_2(1-m)\dot{x}}{1+b_1(1-m)\dot{x}}+\frac{(1-\rho)A-d_1}{a_2}\right]$ with the condition that $d_2>\rho A$.

3.3. Equilibrium Points Stability

3.3.1. Stability of the Equilibrium Point $\boldsymbol{E_0}=(\mathbf{0},\mathbf{0},\mathbf{0})$

The linearization result around $\boldsymbol{E_0}$ produces a Jacobi matrix $\boldsymbol{J}(\boldsymbol{f}(\boldsymbol{E_0}))$which is

$$J(f(\mathrm{E}_0))=\begin{pmatrix}0 & 0 & 0\\ 0 & (1-\rho)A-d_1 & 0\\ 0 & 0 & -(\rho A+d_2)\end{pmatrix}$$

The characteristic equation for $\boldsymbol{J}(\boldsymbol{f}(\boldsymbol{E_0}))$ namely $\boldsymbol{\lambda}\left(\boldsymbol{\lambda}-\left((\mathbf{1}-\boldsymbol{\rho})\boldsymbol{A}-\boldsymbol{d_1}\right)\right)\left(\boldsymbol{\lambda}+(\boldsymbol{\rho A}+\boldsymbol{d_2})\right)=\mathbf{0}$. So that the eigenvalue is obtained $\boldsymbol{\lambda_1}=\mathbf{0}$, $\boldsymbol{\lambda_2}=(\mathbf{1}-\boldsymbol{\rho})\boldsymbol{A}-\boldsymbol{d_1}$, $\boldsymbol{\lambda_3}=-(\boldsymbol{\rho A}+\boldsymbol{d_2})$. Because the eigenvalue $\boldsymbol{\lambda_2}$ is positive, the equilibrium point $\boldsymbol{E_0}$ is unstable.

3.3.2. Stability of the Equilibrium Point $\boldsymbol{E_1}=(\mathbf{1},\mathbf{0},\mathbf{0})$

The linearization result around $\boldsymbol{E_1}$ produces a Jacobi matrix $\boldsymbol{J}(\boldsymbol{f}(\boldsymbol{E_1}))$which is

$$J(f(\mathrm{E}_1)) = \begin{pmatrix} \dfrac{-3-\theta}{(1+\theta)^2} & -\dfrac{a_1(1-m)}{1+b_1(1-m)} & 0 \\ 0 & \dfrac{a_1(1-m)}{1+b_1(1-m)} + (1-\rho)A - d_1 & 0 \\ 0 & 0 & -(\rho A + d_2) \end{pmatrix}$$

The characteristic equation for $\boldsymbol{J(f(E_1))}$ which is $\left(\boldsymbol{\lambda} + \frac{\mathbf{3}+\boldsymbol{\theta}}{(\mathbf{1}+\boldsymbol{\theta})^{\mathbf{2}}}\right)\left(\boldsymbol{\lambda} - \left(\frac{\boldsymbol{a_1(1-m)}}{\boldsymbol{1+b_1(1-m)}} + (\mathbf{1}-\boldsymbol{\rho})\boldsymbol{A} - \boldsymbol{d_1}\right)\right)(\boldsymbol{\lambda} + (\boldsymbol{\rho A} + \boldsymbol{d_2})) = \mathbf{0}$. So that the eigenvalue is obtained $\boldsymbol{\lambda_1} = -\frac{\mathbf{3}+\boldsymbol{\theta}}{(\mathbf{1}+\boldsymbol{\theta})^{\mathbf{2}}}$, $\boldsymbol{\lambda_2} = \left(\frac{\boldsymbol{a_1(1-m)}}{\boldsymbol{1+b_1(1-m)}} + (\mathbf{1}-\boldsymbol{\rho})\boldsymbol{A} - \boldsymbol{d_1}\right)$, $\boldsymbol{\lambda_3} = \left(\frac{\boldsymbol{a_1(1-m)}}{\boldsymbol{1+b_1(1-m)}} + (\mathbf{1}-\boldsymbol{\rho})\boldsymbol{A} - \boldsymbol{d_1}\right)$. For the equilibrium point $\boldsymbol{E_1}$ is locally asymptotically stable, it must be $\boldsymbol{\lambda_2} < \mathbf{0}$. So the equilibrium point $\boldsymbol{E_1}$ is locally asymptotically stable if $\boldsymbol{a_1} < \frac{(\boldsymbol{d_1}-(\mathbf{1}-\boldsymbol{\rho})\boldsymbol{A})(\mathbf{1}+\boldsymbol{b_1}(\mathbf{1}-\boldsymbol{m}))}{(\mathbf{1}-\boldsymbol{m})}$.

3.3.3. Stability of the Equilibrium Point $E_2 = (\bar{x}, \bar{y}, 0)$

The linearization result around $\boldsymbol{E_2}$ produces a Jacobi matrix $\boldsymbol{J(f(E_2))}$Which is

$$J(f(\mathrm{E}_2)) = \begin{pmatrix} \dfrac{-2\bar{x}^3 - 3\theta\bar{x}^2 - \bar{x}^2 + 2\theta\bar{x}}{(\bar{x}+\theta)^2} - \dfrac{a_1(1-m)\bar{x}\bar{y}}{(1+b_1(1-m)\bar{x})^2} & -\dfrac{a_1(1-m)\bar{x}}{1+b_1(1-m)\bar{x}} & 0 \\ \dfrac{a_1(1-m)\bar{x}\bar{y}}{(1+b_1(1-m)\bar{x})^2} & \dfrac{a_1(1-m)\bar{x}}{1+b_1(1-m)\bar{x}} + (1-\rho)A - \dfrac{a_2\bar{z}}{1+b_2\bar{y}} - d_1 & -\dfrac{a_2\bar{y}}{1+b_2\bar{y}} \\ 0 & 0 & \dfrac{a_2\bar{y}}{1+b_2\bar{y}} - \rho A - d_2 \end{pmatrix}$$

$$= \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & g_{23} \\ 0 & 0 & g_{33} \end{pmatrix}$$

The characteristic equation for $\boldsymbol{J(f(E_2))}$ Which is $\boldsymbol{\lambda^3} + (-\boldsymbol{g_{22}} - \boldsymbol{g_{11}} - \boldsymbol{g_{33}})\boldsymbol{\lambda^2} + (\boldsymbol{g_{11}g_{22}} - \boldsymbol{g_{12}g_{21}} + \boldsymbol{g_{33}g_{22}} + \boldsymbol{g_{33}g_{11}})\boldsymbol{\lambda} - \boldsymbol{g_{33}g_{11}g_{22}} + \boldsymbol{g_{33}g_{12}g_{21}} = \mathbf{0}$. Based on the Routh-Hurwitz criterion, it is obtained that the characteristic equation $\boldsymbol{J(f(E_2))}$ will have a negative eigenvalue if and only if qualify the conditions $\boldsymbol{B_3} > \mathbf{0}, \boldsymbol{B_1} > \mathbf{0}$, and $\boldsymbol{B_1B_2} - \boldsymbol{B_3} > \mathbf{0}$. In other words, the equilibrium point $\boldsymbol{E_2}$ is stable if these conditions are satisfied.

3.3.4. Stability of the Equilibrium Point $E_3 = (\dot{x}, \dot{y}, \dot{z})$

The linearization result around $\boldsymbol{E_3}$ produces a Jacobi matrix $\boldsymbol{J(f(E_3))}$Which is

$$J(f(\mathrm{E}_3)) = \begin{pmatrix} \dfrac{-2\dot{x}^3 - 3\theta\dot{x}^2 - \dot{x}^2 + 2\theta\dot{x}}{(\dot{x}+\theta)^2} - \dfrac{a_1(1-m)\dot{x}\dot{y}}{(1+b_1(1-m)\dot{x})^2} & -\dfrac{a_1(1-m)\dot{x}}{1+b_1(1-m)\dot{x}} & 0 \\ \dfrac{a_1(1-m)\dot{x}\dot{y}}{(1+b_1(1-m)\dot{x})^2} & \dfrac{a_1(1-m)\dot{x}}{1+b_1(1-m)\dot{x}} + (1-\rho)A - \dfrac{a_2\dot{z}}{(1+b_2\dot{y})^2} - d_1 & -\dfrac{a_2\dot{y}}{1+b_2\dot{y}} \\ 0 & \dfrac{a_2\dot{z}}{(1+b_2\dot{y})^2} & \dfrac{a_2\dot{y}}{1+b_2\dot{y}} - \rho A - d_2 \end{pmatrix}$$

$$= \begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & h_{23} \\ 0 & h_{32} & h_{33} \end{pmatrix}$$

The characteristic equation for $\boldsymbol{J(f(3))}$ Which is $\boldsymbol{\lambda^3} + (-\boldsymbol{h_{11}} - \boldsymbol{h_{33}} - \boldsymbol{h_{22}})\boldsymbol{\lambda^2} + (\boldsymbol{h_{11}h_{33}} + \boldsymbol{h_{11}h_{22}} - \boldsymbol{h_{12}h_{21}} + \boldsymbol{h_{33}h_{22}} - \boldsymbol{h_{32}h_{23}})\boldsymbol{\lambda} + (\boldsymbol{h_{11}h_{32}h_{23}} - \boldsymbol{h_{11}h_{33}h_{22}} + \boldsymbol{h_{12}h_{21}h_{33}}) = \mathbf{0}$ Based on the Routh-Hurwitz criterion, it is obtained that the characteristic equation $\boldsymbol{J(f(E_3))}$ Will have a negative eigenvalue if and only if it qualify the conditions $\boldsymbol{D_3} > \mathbf{0}, \boldsymbol{D_1} > \mathbf{0}$, and $\boldsymbol{D_1D_2} - \boldsymbol{D_3} > \mathbf{0}$. In other words, the equilibrium point $\boldsymbol{E_3}$ is stable if these conditions are satisfied.

Theorem 2

a) The equilibrium point E_0 is unstable if the eigenvalue λ_2 is positive
b) The equilibrium point E_1 Is locally asymptotically stable if $a_1 < \frac{(d_1-(1-\rho)A)(1+b_1(1-m))}{(1-m)}$
c) The equilibrium point E_2 Is stable if and only if it satisfies the conditions $B_3 > 0, B_1 > 0$, and $B_1B_2 - B_3 > 0$
d) The equilibrium point E_2 Is stable if and only if it satisfies the conditions $D_3 > 0, D_1 > 0$, and $D_1D_2 - D_3 > 0$

3.4. Numerical Simulation

In this section, numerical simulations will be carried out using Maple 18 at each equilibrium point. Simulations for various parameter values in the model are carried out to show the comparison of the results of the analysis of the four equilibrium points. The parameter values used are shown in Table 1 and the initial values used are $\boldsymbol{x(0) = 0.8; y(0) = 0.4; z(0) = 0.2}$.

Table 1 Parameter values used for numerical simulations

Parameters	Value	Source
a_1	5.0	(Nath et al., 2022)
a_2	1.0	(Thirthar et al., 2022)
b_1	3.0	(Nath et al., 2022)
b_2	5.0	Assumption
d_1	0.4	Assumption
d_2	0.1	Assumption
m	0.05	Assumption
θ	0.01	(Nath et al., 2022)
A	0.01	Assumption
ρ	0.08	Assumption

3.4.1. Simulation 1

In this simulation it will be shown that the equilibrium point $\boldsymbol{E_0} = (\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$ with parameter $\boldsymbol{a_1 = 5.0}$, $\boldsymbol{a_2 = 1.0}, \boldsymbol{b_1 = 3.0}, \boldsymbol{b_2 = 5.0}, \boldsymbol{d_1 = 0.4}, \boldsymbol{d_2 = 0.1}, \boldsymbol{m = 0.05}, \boldsymbol{\theta = 0.01}, \boldsymbol{A = 0.01}, \boldsymbol{\rho = 0.08}$ is unstable. Based on Figure 1, the graphic of prey, intermediate predator, and top predator populations are heading towards point 0. This means that there is no interaction between populations in the system.

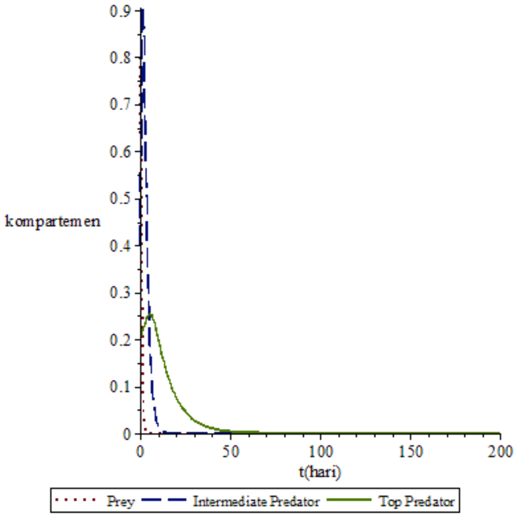

Figure 1 Simulation of System (5) around Equilibrium Point $E_0 = (0,0,0)$

3.4.2. Simulation 2

In this simulation the parameter values used are $\boldsymbol{a_1 = 2.0}$, $\boldsymbol{a_2 = 1.0}$, $\boldsymbol{b_1 = 3.0}$, $\boldsymbol{b_2 = 5.0}$, $\boldsymbol{d_1 = 0.4}$, $\boldsymbol{d_2 = 0.05}, \boldsymbol{m = 0.53}$, $\boldsymbol{\theta = 0.01}$, $\boldsymbol{A = 0.01}$, $\boldsymbol{\rho = 0.08}$. Terms of stability of the equilibrium point $\boldsymbol{E_1} = (\boldsymbol{1}, \boldsymbol{0}, \boldsymbol{0})$ in simulation 2 can be fulfilled, namely $\boldsymbol{a_1} < \frac{(\boldsymbol{d_1}-(1-\boldsymbol{\rho})\boldsymbol{A})(1+\boldsymbol{b_1}(1-\boldsymbol{m}))}{(1-\boldsymbol{m})} \Leftrightarrow \boldsymbol{2.0 < 2.003889362}$. The state of the prey, intermediate predator, and top predator populations in this simulation is shown in Figure 2.

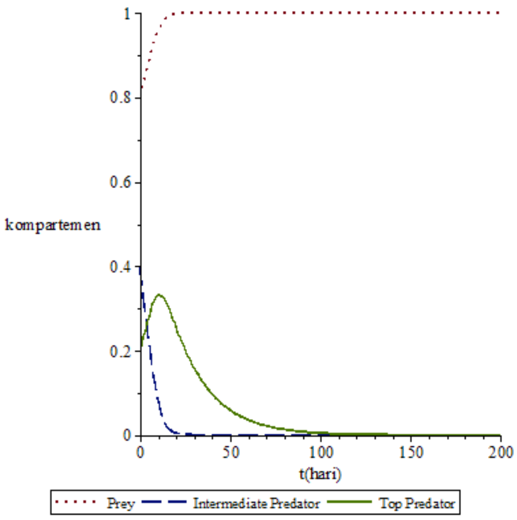

Figure 2 Simulation of System (5) around the Equilibrium Point $E_1 = (1,0,0)$

Based on Figure 2, the graph of prey on the 20th day is increases and stabilizes at point 1, while the intermediate predator graph decreases and stabilizes at point 0. This is caused by changes in parameter values $\boldsymbol{a_1}$ And parameter values $\boldsymbol{m}$ from the previous simulation. Parameter value $\boldsymbol{a_1}$ Which is the maximum predation rate of the intermediate predator population has decreased from $\boldsymbol{a_1} = \mathbf{5.0}$ to $\boldsymbol{a_1} = \mathbf{2.0}$. This means that the limit of intermediate predators in predating on prey *is* decreasing, so that the survival opportunity for the prey population is increasing.

The level of refugia denoted by $\boldsymbol{m}$ also increased from $\boldsymbol{m} = \mathbf{0.05}$ to $\boldsymbol{m} = \mathbf{0.53}$. An increase in the refugia value results in the remaining prey population interacting with the intermediate predator population. In other words, the number of preys protected from predation by intermediate predators increases. Therefore, the graph of the prey population increases and stabilizes at point 1. However, the increase in the value of the refugia level on the intermediate predator population does not have much effect. Although the number of intermediate predators that are protected from predation by top predators is increasing, the opportunity for intermediate predators to prey on prey is decreasing due to the decrease in the parameter value of the refugia level $\boldsymbol{a_1}$. As a result, the intermediate predator population became food deprived and experienced a decline in the graph to stabilize at point 0 on day 20.

Another parameter that has changed in this simulation is the parameter of the natural mortality rate of the top predator population, denoted by $\boldsymbol{d_2}$. The natural mortality rate of the top predator population is getting less, which is from the original $\boldsymbol{d_2} = \mathbf{0.1}$ to $\boldsymbol{d_2} = \mathbf{0.05}$. It can be seen in Figure 1 that the top predator graph goes to point 0 on day 50, while in Figure 2 the top predator graph goes to point 0 on day 100. Then based on Figure 2, when the intermediate predator population is at point 0 or extinction, the graph of the top predator population increases on day 10 to reach the point $\mathbf{0.33}$ due to the presence of alternative food. However, after the top predator population decreased due to the limited supply of alternative food, on the 100th day the top predator population finally became extinct.

3.4.3. Simulation 3

In this simulation, the parameter values used are $\boldsymbol{a_1} = \mathbf{2.0}$, $\boldsymbol{a_2} = \mathbf{1.0}$, $\boldsymbol{b_1} = \mathbf{3.0}$, $\boldsymbol{b_2} = \mathbf{5.0}$, $\boldsymbol{d_1} = \mathbf{0.4}$, $\boldsymbol{d_2} = \mathbf{0.5}$, $\boldsymbol{m} = \mathbf{0.53}$, $\boldsymbol{\theta} = \mathbf{0.01}$, $\boldsymbol{A} = \mathbf{0.02}$, $\boldsymbol{\rho} = \mathbf{0.08}$. The equilibrium point is obtained $\boldsymbol{E_2} = (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \mathbf{0})$ which is $(\mathbf{0.9493859841}; \mathbf{0.1246105157}; \mathbf{0})$. The equilibrium point $\boldsymbol{E_2}$ is stable as shown in Figure 3 by fulfilling the conditions mentioned in Theorem 2.

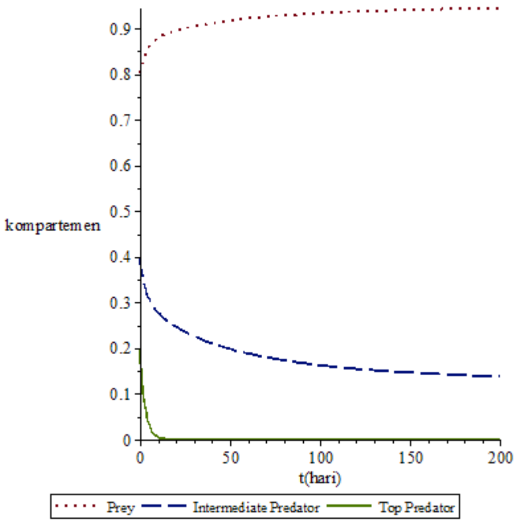

Figure 3 Simulation of System (5) around the Equilibrium Point $E_2 = (\bar{x}, \bar{y}, 0)$

Based on Figure 3, the prey, intermediate predator, and top predator populations are stable near the equilibrium point $\boldsymbol{E_2} = (\mathbf{0.9493859841}; \mathbf{0.1246105157}; \mathbf{0})$. When $\boldsymbol{t} = \mathbf{10}$, the prey population graph increases, while the intermediate predator and top predator population graphs decrease. The prey population graph increases because the level of refugia is quite large, which is equal to $\boldsymbol{m} = \mathbf{0.53}$, so that the prey population is protected from predation by intermediate predators. As a result, the population of intermediate predators that depend on the prey population has decreased. However, some intermediate predator populations survived predation by top predators because of the refugia. The impact of refugia on intermediate predators resulted in a decrease in the population of top predators.

One of the value parameters that changed is the natural mortality rate of the top predator population. If the natural mortality of top predators denoted by $\boldsymbol{d_2}$ Increases, the number of top predator populations living in the system will decrease. Therefore, the top predator population decreases to 0 on day 10 and stabilizes at that point. In other words, the population of top predators that predate on intermediate predator populations or alternative food is getting smaller. This is very beneficial for the intermediate predator population over the alternative food.

Another parameter that also changes is the amount of alternative food availability in the system denoted by $\boldsymbol{A}$. Increasing the value of the parameter $\boldsymbol{A}$ will increase the stable point of the intermediate predator population. If the parameter value $\boldsymbol{A}$ is increased to $\boldsymbol{A} = \mathbf{0.05}$, then the stable point of the intermediate predator population will increase to $\mathbf{0.4414111228}$.

3.4.4. Simulation 4

In this simulation, the parameter values used are $\boldsymbol{a_1} = \mathbf{2.0}$, $\boldsymbol{a_2} = \mathbf{1.0}$, $\boldsymbol{b_1} = \mathbf{3.0}$, $\boldsymbol{b_2} = \mathbf{5.0}$, $\boldsymbol{d_1} = \mathbf{0.4}$, $\boldsymbol{d_2} = \mathbf{0.1}$, $\boldsymbol{m} = \mathbf{0.2}$, $\boldsymbol{\theta} = \mathbf{0.6}$, $\boldsymbol{A} = \mathbf{0.03}$, $\boldsymbol{\rho} = \mathbf{0.08}$.. The equilibrium point is obtained $\boldsymbol{E_3}$ Is at the point $(\dot{\boldsymbol{x}}, \dot{\boldsymbol{y}}, \dot{\boldsymbol{z}})$ Which is $(\mathbf{0.8226723464}; \mathbf{0.1906250000}; \mathbf{0.1369782311})$. The equilibrium point $\boldsymbol{E_3}$ Is stable as shown in Figure 4 by fulfilling the conditions mentioned in Theorem 2.

The parameter that changes in this simulation is the natural mortality rate of the *top predator* population. The parameter $\boldsymbol{d_2}$ decreased by 0.4 to $\boldsymbol{d_2} = \mathbf{0.1}$. This have impact that more of top predators are living and interacting in the system. If the parameter value $\boldsymbol{d_2}$ is decreased again to $\boldsymbol{d_2} = \mathbf{0.02}$, then the resulting graph will be unstable. So, the system will stabilize at $\mathbf{0.02} < \boldsymbol{d_2} \leq \mathbf{0.1}$.

Another parameter that has an effect on this simulation is the parameter $\boldsymbol{m}$ which is the level of refugia. If in this simulation the parameter $\boldsymbol{m}$ continues to use a value of $\boldsymbol{m} = \mathbf{0.53}$, the top predator population will experience extinction on day 80 as shown in Figure 5. The system will stabilize if the parameter value is $\boldsymbol{m}$ parameter value is reduced to $\boldsymbol{m} = \mathbf{0.4}$. If the parameter is set to $\boldsymbol{m} = \mathbf{0.0}$, the system will remain stable. This means that refugia in the system affects the growth of the top predator population. If the value of refugia is higher, the higher the number of prey and intermediate predators that are in the safe zone and remain alive in the system. This results in the top predator population only relying on the availability of alternative food. So, the parameter $\mathbf{m}$ will cause the system to stabilize when $\boldsymbol{m} \leq \mathbf{0.4}$.

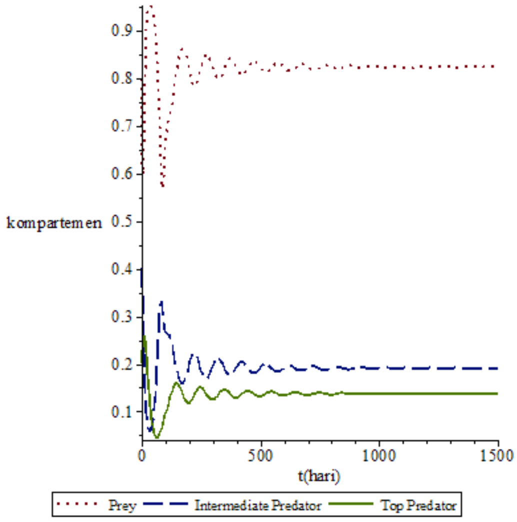

Figure 4 Simulation of System (5) around the Equilibrium Point $E_3 = (\dot{x}, \dot{y}, \dot{z})$

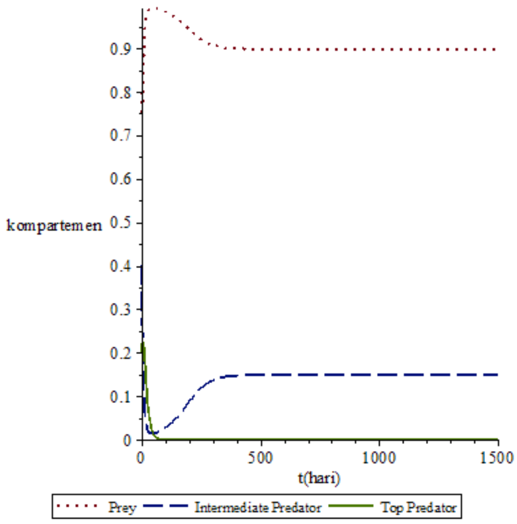

Figure 5 Simulation of System (5) around the Equilibrium Point $E_3 = (\dot{x}, \dot{y}, \dot{z})$ when $m = 0.53$

The level of Allee effect has an influence on the stability of the system when the value of the parameter $\boldsymbol{\theta} \leq \mathbf{0.76}$. Based on Figure 6 when the parameter $\boldsymbol{\theta}$ parameter has a value greater than $\mathbf{0.76}$, then the system will experience chaos and the top predator population will become extinct on day 200. This means that the system will be stable if there is a weak Allee effect ($\boldsymbol{\theta} \leq \mathbf{0.76}$) while there will be chaos if there is a strong Allee effect ($\boldsymbol{\theta} \geq \mathbf{0.76}$). If the weak of Allee effect occurs in the prey, it will make the prey population increase because the natural mortality rate is reduced. The increasing *prey* population will lead to an increase in the intensity of interaction between prey and intermediate predators. This is also commensurate with the growth rate of the top predator population which will also increase.

Parameter *A* which is the level of alternative food availability if the value exceeds $\mathbf{0.1}$ then it will cause the system to be unbalanced. So, the system will be balanced when $\boldsymbol{A} \leq \mathbf{0.1}$. Based on Figure 7, when $\boldsymbol{A} = \mathbf{0.2}$ the top predator population increases on day 15 and then decreases until it reaches point 0 on day 70. The increase in the top predator population is a result of the increase in the value of the parameter $\boldsymbol{A}$ which is the amount of alternative food available. When alternative food is preyed upon by top predators, the intermediate predator population only depends on prey. As a result, the prey population decreased until it approached 0 on day 10, as well as the intermediate predator population whose population decreased until it reached 0 when $\boldsymbol{t} = \mathbf{20}$. When the prey and intermediate predator populations experience extinction, the *top predator* population can still survive because of alternative food. Then when the intermediate predator is extinct, there is no one to prey on the prey so that the *prey* population grows again.

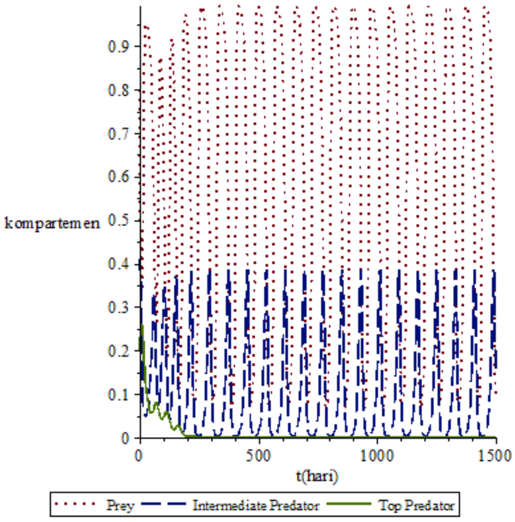

Figure 6 Simulation of System (5) around the Equilibrium Point $E_3 = (\dot{x}, \dot{y}, \dot{z})$ when $\theta = 0.77$

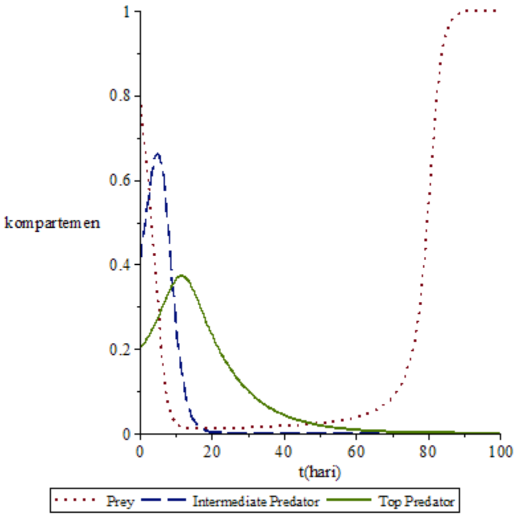

Figure 7 Simulation of System (5) around the Equilibrium Point $E_3 = (\dot{x}, \dot{y}, \dot{z})$ when $A = 0.2$

Parameters $\boldsymbol{\rho}$ is part of the alternative food that is preyed upon by the top predator. So, when $\boldsymbol{\rho}$ is $\mathbf{0.08}$, it means that the top predator population only consumes alternative food by the amount of $\mathbf{0.08}$. While the remaining part, namely $(\mathbf{1}-\boldsymbol{\rho})\boldsymbol{A}$ will be consumed by intermediate predators. So, when $\boldsymbol{\rho} = \mathbf{0.0}$, then all alternative food is only consumed by intermediate predators, causing the top predator population to experience extinction. In contrast, if the parameter $\boldsymbol{\rho} = \mathbf{1.0}$, then the intermediate predator population does not get the opportunity to consume alternative food. As a result, the intermediate predator population is extinct. So, to keep the system stable, the parameter $\boldsymbol{\rho}$ must be $\mathbf{0.1} \leq \boldsymbol{\rho} \leq \mathbf{0.5}$.

4. Conclusion

Based on the results of the analysis discussed in the previous chapter, the system is stable at the equilibrium point $\boldsymbol{E_3} = (\dot{\boldsymbol{x}}, \dot{\boldsymbol{y}}, \dot{\boldsymbol{z}})$. The parameters Allee effect, refugia, and alternative food have an influence on the growth of each population. The level of the Allee effect has an influence on the stability of the system when the value is $\boldsymbol{\theta} \leq \mathbf{0.76}$. If the parameter $\boldsymbol{\theta}$ parameter has a value greater than $\mathbf{0.76}$, then the system will experience chaos and the top predator population will become extinct on day 200. Then *refugia* in the system affects the growth of the top predator population. If the value of refugia is higher, the higher the number of prey and intermediate predators that are in the safe zone and remain alive in the system. The top predator population experiences extinction on day 80 if the refugia value is higher. $\boldsymbol{m} > \mathbf{0.4}$. So, the parameter $\boldsymbol{m}$ will cause the system to stabilize when $\boldsymbol{m} \leq \mathbf{0.4}$. Then the parameter *A* which is the level of availability of alternative food will cause the system to be balanced when $\boldsymbol{A} \leq \mathbf{0.1}$. At the time of $\boldsymbol{A} > \mathbf{0.1}$, the population of top predators increases on day 15 and then decreases until it reaches point 0 on day 70. Likewise, the population of intermediate predators whose population decreases until it reaches point 0 when $t = 20$. So, the system will experience an imbalance if $A > 0.1$. It can be concluded that refugia and alternative food minimally affect the top predator's population growth, while the Allee effect influences the growth of prey and intermediate predator populations.

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